Half-integrality of node-capacitated multiflows and tree-shaped facility locations on trees

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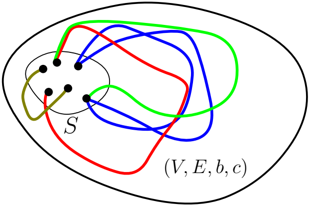
(V, E, S, b, c): network
(V, E): undirected graph
$S \subseteq V$: terminal set
$b : V \to \mathbf{Z}_+$: node-capacity
$c : E \to \mathbf{Z}_+$: edge-capacity



Definition

Multiflow $f = (\mathcal{P}, \lambda) \overset{\text{def}}{\Longleftrightarrow}$
$\mathcal{P}$: a set of S-paths & $\lambda : \mathcal{P} \to \mathbf{R}_+$: a flow-value function satisfying

$$\sum \{\lambda(P) \mid P \in \mathcal{P} : x \in VP\} \leq b(x) \quad (x \in V),$$
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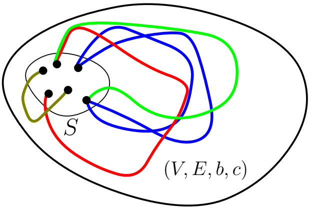

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routing in networks, VLSI-layout, disjoint paths,
LP-relaxations of NP-hard problems (multicut, 0-extension, ...)

Our Problem

(V, E, S, b, c): network, $\mu : \binom{S}{2} \to \mathbf{R}_+$

Definition (Flow-value of $f = (\mathcal{P}, \lambda)$)

$$\mathrm{val}(\mu, f) := \sum \{\lambda(P)\mu(s_P, t_P) \mid P \in \mathcal{P}\}.$$

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$S = \{s, t\} \Rightarrow$ single commodity flow (Ford-Fulkerson 56)
$\mu = 1 \Rightarrow$ free multiflow (Lovász 76, Cherkassky 77; Vazirani 01, Pap 07, Babenko-Karzanov 08)

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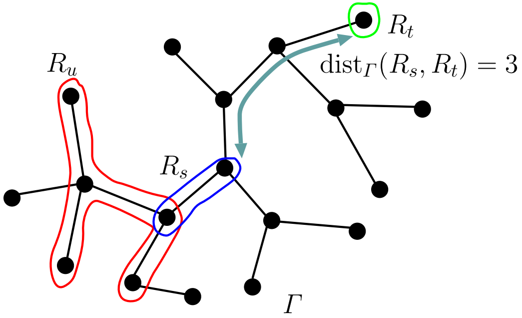
We are interested in the behavior of multiflows for fixed μ and an arbitrary network (V, E, S, b, c)

Subtree distance (H. 06)

Definition

$\mu : \binom{S}{2} \to \mathbf{R}_+$ is a *subtree distance* $\overset{\text{def}}{\Longleftrightarrow}$
$\exists$ tree Γ, $\alpha \in \mathbf{R}_{>0}$, a family $\{R_s \mid s \in S\}$ of subtrees s.t.

$$\mu(s,t) = \alpha \operatorname{dist}_\Gamma(R_s, R_t) \quad (s,t \in S).$$



Main Theorem (combinatorial min-max relation)

(V, E, S, b, c), μ: subtree distance realized by $(\Gamma, \alpha; \{R_s\}_{s \in S})$

Theorem (H. 10)

$$\max_f \mathrm{val}(\mu, f)$$
$$= \alpha \min \sum_{x \in V} b(x) \mathrm{diam} F(x) + \sum_{xy \in E} c(xy) \mathrm{dist}_\Gamma(F(x), F(y))$$
$$\text{s.t. } F : V \to \mathcal{F}\Gamma \text{ (all subtrees)}, \ F(s) \cap R_s \neq \emptyset \ (s \in S).$$

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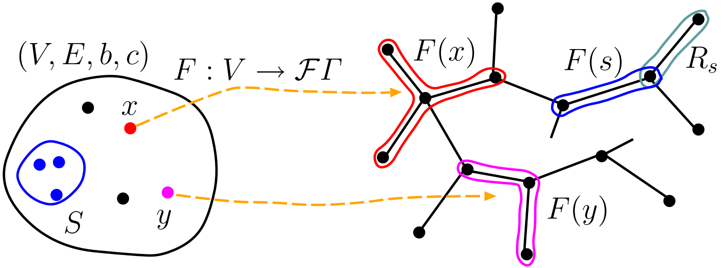

Main Theorem (half-integrality & polytime algorithm)

Theorem (H. 10, cond.)

- There exists a *half-integral* μ-max multiflow.
- There exists a strongly polytime algorithm to find a half-integral μ-max multiflow and an optimal subtree location.

Remark

- Tree-shaped facility location (Mineaka 85, Lowe-Tamir 92, Hakimi-Schmeichel-Labbe 93, ...)
- $\mu \neq$ tree distance $\Rightarrow \forall k, \exists (V, E, S, b, c), \nexists\ 1/k$-integral μ-max multiflow

When $b \to \infty$ (edge-only-capacitated)

$$\max_f \mathrm{val}(\mu, f)$$
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&= \alpha \min \sum_{xy \in E} c(xy) \mathrm{dist}_\Gamma (\rho(x), \rho(y)) \\
&\qquad \text{s.t. } \rho : V \to V\Gamma,\ \rho(s) \in R_s\ (s \in S)
\end{aligned}
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$\Rightarrow$ point location on tree Γ

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Interpretation ?

$V_t := \{x \in V \mid t \in F(x)\} \ (t \in V\Gamma)$.

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$V_t := \{x \in V \mid t \in F(x)\}$ $(t \in V\Gamma)$.
$(\Gamma, \{V_t\}_{t \in V\Gamma})$: *tree-decomposition* of (V, E).

Example 1 (node-only capacitated; $c \to \infty$, $b|_S \to \infty$)

$S = \{s, t\}$, $\mu(s, t) = 1$, $\Gamma = v_s v_t$, $R_s = \{v_s\}$, $R_t = \{v_t\}$

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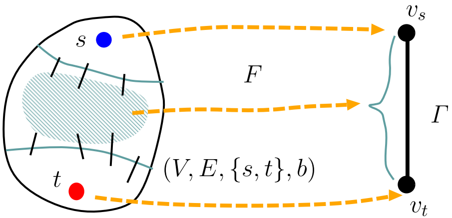
$$\max_f \mathrm{val}(\mu, f) \quad = \quad \min \textstyle\sum_{x \in V \setminus S} b(x) \mathrm{diam} F(x)$$

$$\text{s.t.} \quad F(x) = \{v_s\}, \{v_t\}, \text{ or } \{v_s, v_t\} \quad (x \in V),$$

$$F(x) \cap F(y) \neq \emptyset \qquad (xy \in E),$$

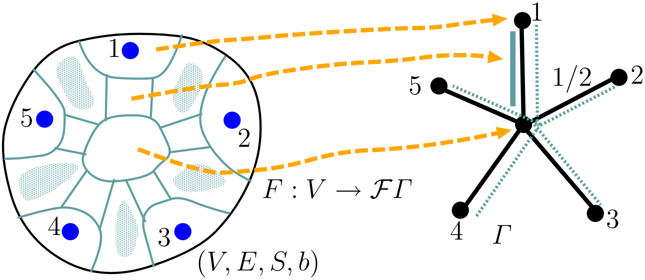
$$(F(s), F(t)) = (\{v_s\}, \{v_t\}).$$



$\to$ Menger's theorem

Example 2 (node-only capacitated; $c \to \infty$, $b|_S \to \infty$)

$S = \{s_1, s_2, \ldots, s_k\}$, $\mu = 1$



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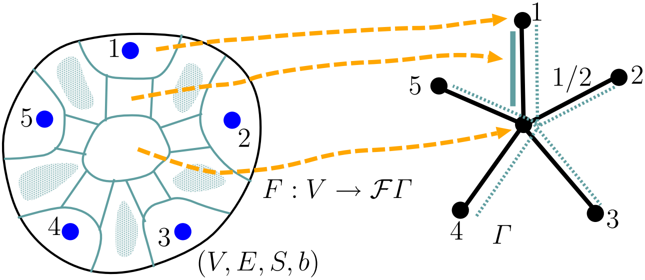

$$\max_f \operatorname{val}(\mu, f) \quad = \quad \min \; b(U_0) + \frac{1}{2}\sum_{i=1}^{k} b(\mathrm{Bd}(U_i))$$

$$\text{s.t.} \quad U_0, U_1, U_2, \ldots, U_k (\text{disjoint}), \; s_i \in U_i.$$

cf. Vazirani 01, Mader 78.

Proof Sketch

Proof consists of two parts:

Duality relation: LP-dualilty & subtree lemma

Half-integrality: optimality criterion & fractional b-matching

Our proof is constructive ($\to$ polynomial time algorithm)
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Our proof is constructive ($\to$ polynomial time algorithm)
but.. *is not combinatorial* by the use of generic LP-solver.

LP-duality & Subtree Lemma

$$\max_f \mathrm{val}(\mu, f) \quad = \quad \min \textstyle\sum_{x \in V} b(x)h(x) + \sum_{xy \in E} c(xy)d(xy)$$

$$\begin{aligned} \text{s.t. } & d(xy) + d(yz) - d(xz) + h(y) \geq 0 \ (x, y, z \in V), \\ & d(st) + h(s) + h(t) \geq \mu(s,t) \qquad (s, t \in S), \\ & h : V \to \mathbf{R}_+, \ d : \tbinom{V}{2} \to \mathbf{R}_+. \end{aligned}$$

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Suppose μ is realized by $(\Gamma, 1; \{R_s\}_{s\in S})$.
$\overline{\Gamma} \subseteq \mathbf{R}^2$: a *geometric realization* of Γ
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(2) $\forall (h, d)$ feasible to LP, $\exists F : V \to \mathcal{F}\overline{\Gamma}$, $R_s \cap F(s) \neq \emptyset$,

$$\mathrm{dist}_{\Gamma}(F(x), F(y)) \;\leq\; d(xy) \quad (x, y \in V),$$
$$\mathrm{diam} F(x) \;\leq\; h(x) \quad (x \in V).$$

Proof of Subtree Lemma (for easiest case; $R_s = \{v_s\}$, $b \to \infty$, $h \to 0$)

Given $d : \binom{V}{2} \to \mathbf{R}_+$ s.t. $d(xy) + d(yz) - d(xz) \geq 0$, $d|_S \geq \mu$.

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$$V = \{\overbrace{x_1, x_2, \ldots, x_k}^{S}, x_{k+1}, x_{k+2}, \ldots, x_n\}$$

$$\rho(x_i) := \begin{cases} v_{x_i} & (i = 1, 2, \ldots, k) \\ \text{any point in} \bigcap_{j=1}^{i-1} \mathrm{Ball}(\rho(x_j), d(x_j x_i)) & (i = k+1, k+2, \ldots) \end{cases}$$

Claim: $\bigcap \mathrm{Ball}$ is nonempty

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$\Leftarrow \mathrm{Ball}(\rho(x_j), d(x_j x_i)) \cap \mathrm{Ball}(\rho(x_k), d(x_k x_i)) \neq \emptyset$ $(\forall j, k)$ *(Helly)*

$\Leftarrow d(x_j x_i) + d(x_i x_k) \geq d(x_j, x_k) \geq \mathrm{dist}_\Gamma(\rho(x_j), \rho(x_k))$

cf. Aronszajn-Panitchpakdi 56

Minimum cost multiflows (node-only-capacitated; $c \to \infty$, $b|_S \to \infty$)

$a : V \to \mathbf{R}_+$: node-cost
$\mathrm{cost}(a, f) := \sum\{\lambda(P)a(VP) \mid P \in \mathcal{P}\}$

Problem (mincost multiflow)

Maximize $\mathrm{val}(\mu, f) - \mathrm{cost}(a, f)$ *over all multiflows* f.

Theorem (H. 10)

If μ *is a subtree distance, then there exists a half-integral mincost multiflow.*

- edge-only-capacitated & $\mu = 1$ (Karzanov 79, 94)
- node-capacitated & $\mu = 1$ (Pap 08, Babenko-Karzanov 09)

Proposition

$$\max_{f} \mathrm{val}(\mu, f) - \mathrm{cost}(a, f)$$

$$= \min \sum_{y \in V \setminus S} b(y) \max\{0, \mathrm{diam} F(y) - a(y)\}$$

$$\text{s.t. } F : V \to \mathcal{F}\overline{\Gamma},$$

$$F(x) \cap F(y) \neq \emptyset \quad (xy \in E),$$

$$F(s) \text{ is a single point in } R_s \quad (s \in S).$$

$\to$ optimality criterion (kilta condition)

Flow-support $\zeta^f : E \to \mathbf{R}_+$

$$\zeta^f(e) = \sum \{\lambda(P) \mid P \in \mathcal{P} : e \in P\} \quad (e \in E)$$

Polyhedron of optimal-flow-suppposts:

$$\mathcal{P}^* := \{\zeta : E \to \mathbf{R}_+ \mid \zeta = \zeta^{f^*} \text{ for some optimal multiflow } f^*\}$$

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Proposition

(1) Given an optimal subtree map $F^* : V \to \mathcal{F}\overline{\Gamma}$, we can obtain polynomial size linear inequality description of $\mathcal{P}^*$.

(2) $\mathcal{P}^*$ is half-integral.

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Proposition

Given a half-integral extreme point ζ^* in $\mathcal{P}^*$, we can construct a half-integral optimal multiflow f^* with $\zeta^* = \zeta^{f^*}$ in polytime.

Optimality criterion

$\delta(y)$: the set of edges incident to y

Lemma

$f = (\mathcal{P}, \lambda)$ and $F : V \to \mathcal{F}\overline{\Gamma}$ are both optimal $\Leftrightarrow$

(1) $$\zeta^f(\delta y) = \begin{cases} 2b(y) & \text{if } \mathrm{diam} F(y) > a(y) \\ 0 & \text{if } \mathrm{diam} F(y) < a(y) \end{cases} \quad (\forall y \in V \setminus S).$$

(2) $\forall P \in \mathcal{P}$, $\mathrm{diam} F(VP) = \mathrm{dist}_\Gamma(R_{s_P}, R_{t_P})$.

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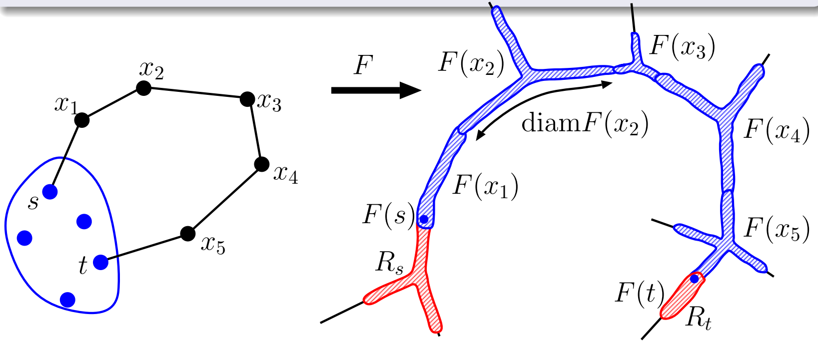

Concluding remarks

- Combinatorial polynomial time algorithm (*Challenge !!*)
- Convex-cost multiflows (Fenchel duality theory)
- *What is discrete convexity theory for multiflows ?*
- Weighted version of Mader's S-paths packing ?
  ($\mu \neq$ subtree distance $\Rightarrow$ μ-max integer multiflow is NP-hard)