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T_X-Approaches to Multiflows and Metrics

By

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Abstract

This paper is an exposition of a unified approach to multiflow problems using certain polyhedral objects called *tight spans* or T_X*-spaces*. The tight span was introduced by Isbell and Dress, independent on the multiflow research. In the middle of 90's, Karzanov and Chepoi explored the significance of tight spans in the multiflow theory. We explain how the tight span derives min-max relations to multiflow problems and how its geometry affects discreteness issues of flows and potentials.

§ 1. Introduction

This paper is an exposition of a unified approach to multiflow problems using certain polyhedral objects called *tight spans* or T_X*-spaces*. The tight span was introduced by Isbell [17] and Dress [7] for a metric, independent on the multiflow research. By a *metric* μ on a set S we mean a function defined $S \times S$ satisfying $\mu(s,t) = \mu(t,s) \geq \mu(s,s) = 0$ and the triangle inequality $\mu(s,t) + \mu(t,u) \geq \mu(s,t)$ for $s,t,u \in S$. The *tight span* T_μ for μ is defined to be the set of (pointwise) minimal elements of the unbounded polyhedron

$$P_\mu = \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}. \tag{1.1}$$

Although a duality relationship between multiflows and metrics was known in 70's [16, 28], it was the middle of 90's when the significance of tight spans in the multiflow theory was revealed by Karzanov [22, 23] and Chepoi [3]. Recently, the author [10] considered the tight span of a *possibly nonmetric* distance μ, where by a *distance* μ we mean a function on $S \times S$ satisfying only $\mu(s,t) = \mu(t,s) \geq \mu(s,s) = 0$ for $s,t \in S$. The subsequent paper [11] showed that nonmetric tight spans provide general combinatorial

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duality relations for weighted maximum multiflow problems, unifying previously known results for 0-1-weighted and metric-weighted maximum multiflow problems [19, 23]. In this paper, following [11] (and [12]), we explain how the tight span T_μ derives such a combinatorial min-max relation in multiflow problems and how geometry of T_μ affects discreteness issues of flows and potentials.

We begin with introducing basic notions. Let G be an undirected graph with nonnegative edge capacity $c : EG \to \mathbf{R}_+$. Let $S \subseteq VG$ be the set of terminals. A set $\mathcal{P}$ of paths in G whose ends belong to S together with nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ is called a *multiflow* of (G, c) if it satisfies the capacity constraint

$$\sum_{P \in \mathcal{P} : e \in P} \lambda(P) \leq c(e) \quad (e \in EG). \tag{1.2}$$

This paper mainly deals with the following multiflow maximization problems. For a distance μ on terminals S, the *μ-weighted maximum multiflow problem* (*μ-problem*) is formulated as follows:

$$\text{Maximize} \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \text{ over multiflow } (\mathcal{P}, \lambda) \text{ for } (G, c), \tag{1.3}$$

where s_P, t_P are the ends of P. μ-problems contain several basic multiflow problems. In particular, the 0-1 case is of particular combinatorial interest. In this case, 0-1 distance μ is regarded as the commodity graph, and μ-problem is the problem of maximizing the total sum of multiflows connecting pairs of terminals specified by $\mu(s, t) = 1$. For example, the case where S is a 2-set $\{s, t\}$ with $\mu(s, t) = 1$ corresponds to the single-commodity flow problem. The case where S is a 4-set $\{s, t, s', t'\}$ and $\mu(s, t) = \mu(s', t') = 1$ and others are zero corresponds to the two-commodity flow problem. The case of $\mu(s, t) = 1$ for all distinct $s, t \in S$ is called the *free multiflow problem.* In the three cases above, there are combinatorial duality relations: Ford-Fulkerson's max-flow min-cut theorem [8], Hu's max-biflow min-cut theorem [14], and Lovász-Cherkassky's duality theorem [4, 26].

A T_X-approach we would like to describe here gives a unified derivation to such duality relations. The core of this approach is to consider the following *continuous location problem* on the tight span T_μ endowed with the l_∞-distance:

$$\begin{aligned} \text{Minimize} \quad & \sum_{xy \in EG} c(xy)\|\rho(x) - \rho(y)\|_\infty \\ \text{subject to} \quad & \rho : VG \to T_\mu, \\ & \rho(s) \in T_{\mu,s} \ (s \in S), \end{aligned} \tag{1.4}$$

where the sets $T_\mu, T_{\mu,s} \subseteq \mathbf{R}^S$ are defined by

$$P_\mu = \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}, \tag{1.5}$$

$$T_\mu = \text{the set of minimal elements of } P_\mu, \tag{1.6}$$

$$T_{\mu,s} = \{p \in T_\mu \mid p(s) = 0\} \quad (s \in S). \tag{1.7}$$

We call the problem (1.4) the *tight-span-dual* (*T-dual*) to μ-problem. In analogy to the network flow theory, we call ρ in T-dual (1.4) a *potential*. T-dual is indeed a dual of μ-problem as follows.

Theorem 1.1 ([11]). *The maximum value of μ-problem is equal to the minimum value of T-dual.*

Furthermore, if $\dim T_\mu \leq 2$, then T-dual can be *discretized* as follows.

Theorem 1.2 ([11]). *If μ is a rational distance with $\dim T_\mu \leq 2$, then there exists a finite set Z in T_μ such that for every graph (G,c) with $S \subseteq VG$ we can take an optimal potential ρ in T-dual with $\rho(VG) \subseteq Z$.*

Namely, if $\dim T_\mu \leq 2$, then T-dual is reduced to the following *discrete* location problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{xy \in EG} c(xy) \|\rho(x) - \rho(y)\|_\infty \\ \text{subject to} \quad & \rho : VG \to T_\mu \cap Z, \\ & \rho(s) \in T_{\mu,s} \cap Z \ (s \in S). \end{aligned} \tag{1.8}$$

This might be regarded as an analogous phenomenon of the *discreteness of potential* in the network flow theory, and gives a general combinatorial duality relation for μ-problems with $\dim T_\mu \leq 2$, which includes max-flow min-cut theorem, max-biflow min-cut theorem, and so on. This duality relation was recognized in the case of metrics μ [22, 23]. The main contribution of [11] is to extended it to general distances μ.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we illustrate several examples of T-dual to explain how T-dual derives combinatorial duality relations. In Section 4, we give a sketch of the proof of Theorem 1.2 with emphasis on a geometric intuition. In Section 5, we explain an application of tight spans to metric packing problems, which are dual to multiflow feasibility problems. In Section 6, we describe related issues, future research directions, and open questions.

Notation $\mathbf{R}$, $\mathbf{R}_+$, and $\mathbf{Z}$ denote the sets of reals, nonnegative reals, and integers, respectively. For a set S, the characteristic vector χ_S is defined as: $\chi_S(s) = 1$ for

$s \in S$ and $\chi_S(t) = 0$ for $t \notin S$. We simply denote $\chi_{\{s\}}$ by χ_s. For a graph G, VG and EG denote the sets of vertices and edges of G, respectively. For a graph G, dist_G is the graph metric on VG induced by G. A subgraph G' of G is called *isometric* if $\mathrm{dist}_{G'}(x,y) = \mathrm{dist}_G(x,y)$ for $x, y \in VG'$. We use the basic terminology in the polytope theory, such as *faces*, *extreme points*, *polyhedral subdivisions*; see [33]. We call a k-dimensional face a *k-face*.

§ 2. T-dual to maximum multiflow problems

In this section, we prove Theorem 1.1. It is known that the LP-dual of μ-problem is given by the following:

$$\begin{aligned} \text{(2.1)} \qquad \text{Minimize} \quad & \sum_{xy \in EG} c(xy) d(x,y) \\ \text{subject to} \quad & d : \text{metric on } VG, \\ & d(s,t) \geq \mu(s,t) \ (s,t \in S). \end{aligned}$$

This is a variant of the so-called Japanese theorem due to Onaga and Kakusho [28] and Iri [16]. We show that (2.1) is reduced to T-dual (1.4). The proof consists of two lemmas. The first lemma states that LP-dual (2.1) is reduced to a location problem on P_μ.

Lemma 2.1. *The optimal value of μ-problem is equal to the minimum value of the following problem:*

$$\begin{aligned} \text{(2.2)} \qquad \text{Minimize} \quad & \sum_{xy \in EG} c(xy) \|\rho(x) - \rho(y)\|_\infty \\ \text{subject to} \quad & \rho : VG \to P_\mu, \\ & \rho(s) \in P_{\mu,s} \ (s \in S), \end{aligned}$$

where the subset $P_{\mu,s} \subseteq P_\mu$ for $s \in S$ is defined by

$$P_{\mu,s} = \{p \in P_\mu \mid p(s) = 0\}. \tag{2.3}$$

Proof. For $\rho : VG \to P_\mu$ with $\rho(s) \in P_{\mu,s}$ $(s \in S)$, define a metric d^ρ on VG by

$$d^\rho(x,y) = \|\rho(x) - \rho(y)\|_\infty \quad (x, y \in VG). \tag{2.4}$$

Then we easily see that d^ρ is feasible to (2.1). Conversely, take a metric d feasible to (2.1). Define a map $\rho^d : VG \to \mathbf{R}^S$ by

$$(\rho^d(x))(s) = d(s,x) \quad (s \in S, x \in VG). \tag{2.5}$$

By definition of $\rho^d(x)$ and the triangle inequality, ρ^d is feasible to (2.2). Furthermore, the triangle inequality $d(x,y) \geq |d(x,s) - d(s,y)|$ implies $d(x,y) \geq \|\rho^d(x) - \rho^d(y)\|_\infty$. The nonnegativity of c implies

$$\sum_{xy \in EG} c(xy)d(x,y) \geq \sum_{xy \in EG} c(xy)\|\rho^d(x) - \rho^d(y)\|_\infty. \tag{2.6}$$

Thus we can always take an optimal solution of (2.1) as d^ρ for some potential ρ in (1.4). □

The second lemma, due to Dress, states the existence of a nonexpansive retraction from P_μ to T_μ.

Lemma 2.2 ([7, p.331,(1.9)]). *There is a map $\phi : P_\mu \to T_\mu$ such that*

(1) $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ *for $p, q \in P_\mu$, and*

(2) $\phi(p) \leq p$ *for $p \in P_\mu$, and thus ϕ is identical on T_μ.*

Sketch of the proof. For $s \in S$, define $\phi_s : P_\mu \to P_\mu$ by

$$\phi_s(p) := p - \chi_s \max\{\epsilon \geq 0 \mid p - \epsilon\chi_s \in P_\mu\}. \tag{2.7}$$

Then, one can show that ϕ_s satisfies (1-2). Let $S = \{s_1, s_2, \ldots, s_m\}$. Then the map

$$\phi_{s_m} \circ \phi_{s_{m-1}} \circ \cdots \circ \phi_{s_1} \tag{2.8}$$

is a desired one. □

Since c is nonnegative, by Lemma 2.2, we can always take an optimal solution of (2.2) from potentials in T-dual (1.4). Thus we obtain Theorem 1.1. By the proof of Lemma 2.1, the relationship between LP-dual (2.2) and T-dual (1.4) is given as follows:

- For a metric d *minimal* in the feasible region of LP-dual (2.2), the map ρ^d defined by

  $$(\rho^d(x))(s) = d(s,x) \quad (x \in VG, s \in S) \tag{2.9}$$

  is a potential in T-dual (1.4).

- For a potential ρ in T-dual (1.4), a metric d^ρ defined by

  $$d^\rho(x,y) = \|\rho(x) - \rho(y)\|_\infty \quad (x, y \in VG) \tag{2.10}$$

  is feasible to LP-dual (2.2).

We end this section with listing basic properties of T_μ.

Lemma 2.3.

(1) *(T_μ, l_∞) is geodesic, i.e., for $p, q \in T_\mu$ there is a path in T_μ of length $\|p-q\|_\infty$.*

(2) *$\mu(s,t) = \inf\{\|p-q\|_\infty \mid p \in T_{\mu,s}, q \in T_{\mu,t}\}$ for $s, t \in S$.*

(3) *If μ is a metric, then $T_{\mu,s}$ is a single point μ_s defined by*

$$\mu_s(t) = \mu(s,t) \quad (t \in S), \tag{2.11}$$

i.e., μ_s is the s-th row vector of μ.

(4) *$p(s) = \inf\{\|p-q\|_\infty \mid q \in T_{\mu,s}\}$ for $p \in T_\mu$ and $s \in S$.*

(5) *$|p(s) - p(t)| \leq \mu(s,t)$ for $p \in T_\mu$ and $s, t \in S$.*

(1) is an easy corollary of Lemma 2.2. (2-3) were shown in [10] and (4) was shown in [11]. (4) follows from (1) and (3). They are extensions of [7, Theorem 2 (i), (ii), and (iv)]. The properties (2-3) mean that μ is isometrically embedded into T_μ as the l_1-distance among subsets $T_{\mu,s}$.

§ 3. Some examples

In this section, we explain how T-duals derive combinatorial duality relations in multiflow maximization problems. The message of this section is:

The shape of T_μ is a min-max formula of μ-problem.

§ 3.1. Single commodity flows

The first example is well-known single commodity flow problem. In this case, the terminal S is a 2-set $\{s,t\}$, and $\mu(s,t) = 1$. Therefore, P_μ is an unbounded polyhedron in the plane, T_μ is the segment, and $T_{\mu,s} = \{\chi_t\}$ and $T_{\mu,t} = \{\chi_s\}$ are the endpoints of the segment; see Figure 1 (a). Therefore, T-dual is equivalent to the following problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{xy \in EG} c(xy)|\rho(x) - \rho(y)| \\ \text{subject to} \quad & \rho : VG \to [0,1], \ \rho(s) = 0, \rho(t) = 1. \end{aligned} \tag{3.1}$$

Namely, ρ is an ordinary (scalar) potential. This problem can be *discretized* into the minimum cut problem as follows. We can easily see that for any map $\rho : VG \to [0,1]$,

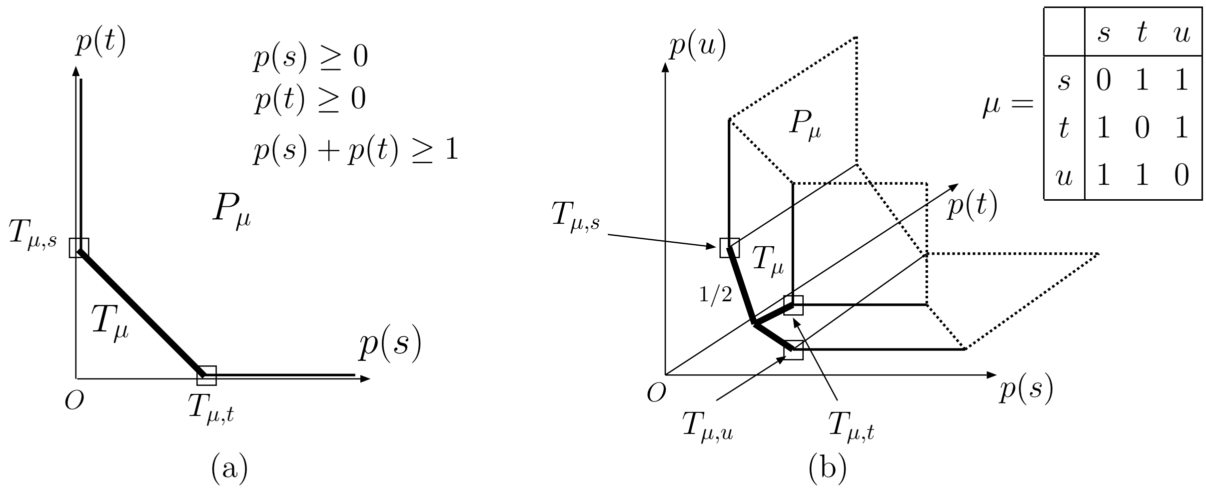

Figure 1. (a) T_μ of a 2-point distance and (b) T_μ of all-one 3-point distance

the corresponding metric d^ρ can be represented as a convex combination of $\{d^{\rho_i}\}_{i \in I}$ for maps $\rho_i : V \to \{0, 1\}$. Therefore, (3.1) is discretized into

$$\text{(3.2)} \qquad \begin{aligned} &\text{Minimize} \quad \sum_{xy \in EG} c(xy)|\rho(x) - \rho(y)| \\ &\text{subject to} \quad \rho : VG \to \{0, 1\}, \ \rho(s) = 0, \rho(t) = 1. \end{aligned}$$

This is nothing but the minimum cut problem. Then we obtain Ford-Fulkerson's max-flow min-cut theorem (without integrality of optimal flows).

§ 3.2. Free multiflows

The second example we consider is the *free multiflow problem.* The corresponding distance μ is all-one distance, that is, $\mu(s, t) = 1$ for distinct $s, t \in S$. Then T_μ is a star having the center $1/2\chi_S$, $\#S$ leafs $T_{\mu,s} = \{\chi_{S \setminus s}\}$ $(s \in S)$, and edge length $1/2$; see Figure 1 (b) for the three terminal case. By the argument similar to the previous single flow example, T-dual is discretized into the *discrete location problem on the star* as follows. Let Γ be the graph of 1-skeleton of T_μ, which is the star with the center $p^O = 1/2\chi_S$ and the leafs $p^s = \chi_{S \setminus s}$ $(s \in S)$. Then T-dual is reduced to:

$$\text{(3.3)} \qquad \begin{aligned} &\text{Minimize} \quad \frac{1}{2} \sum_{xy \in EG} c(xy) \mathrm{dist}_\Gamma(x, y) \\ &\text{subject to} \quad \rho : VG \to V\Gamma, \ \rho(s) = p^s \ (s \in S). \end{aligned}$$

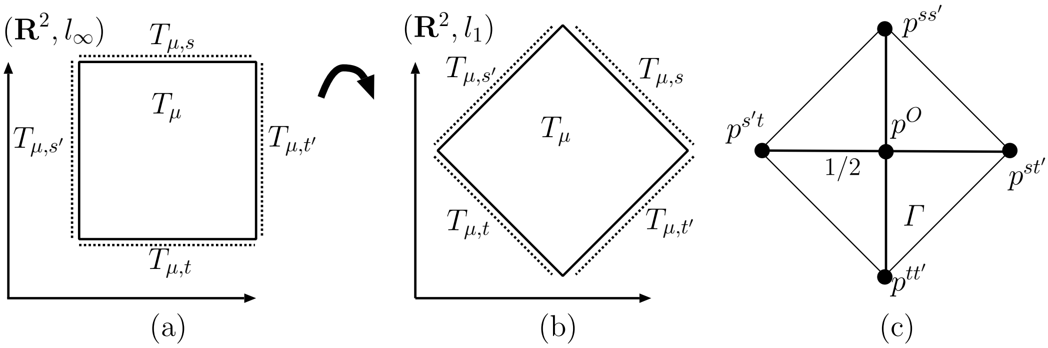

Figure 2. Two-commodity tight span

From this, we obtain Lovász-Cherkassky duality relation (without half-integrality of optimal flows):

$$\text{the optimal multiflow value} = \frac{1}{2}\sum_{t \in S}\{t\text{-}S \setminus t \text{ minimum cut value}\}. \tag{3.4}$$

§ 3.3. Two-commodity flows

The third example is the two-commodity flow maximization problem. This case corresponds to: S is a 4-set $\{s, s', t, t'\}$, and $\mu(s,t) = \mu(s',t') = 1$ and others are zero. Then P_μ is a 4-dimensional polyhedron, and thus we cannot draw it. Its minimal element T_μ, however, is 2-dimensional. Indeed, an easy calculation shows

$$T_\mu = 1/2\chi_S + \{a(\chi_s - \chi_t) + b(\chi_{s'} - \chi_{t'}) \mid -1/2 \le a, b \le 1/2\}. \tag{3.5}$$

Therefore T_μ is isomorphic to the unit square in the l_∞-plane $(\mathbf{R}^2, l_\infty)$ by projection to $\mathbf{R}^{\{s,s'\}}$, and $T_{\mu,s}$ $(s \in S)$ are its four edges; see Figure 2 (a). We show that T-dual in this case is also discretized. Recall the well-known fact that the l_∞-plane $(\mathbf{R}^2, l_\infty)$ is isomorphic to the l_1-plane $(\mathbf{R}^2, l_1)$ by 45 degree location

$$(x_1, x_2) \mapsto \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}\right). \tag{3.6}$$

By the map, T_μ is isomorphic to the square in the l_1-plane; see Figure 2 (b). We can subdivide T_μ into four isosceles right triangles with its shorter edges parallel to l_1-axes as in Figure 2 (c), where l_1-axes mean vectors $(1,1)$ and $(1,-1)$ in $(\mathbf{R}^2, l_\infty)$ or $(0,1)$ and $(1,0)$ in $(\mathbf{R}^2, l_1)$. This subdivision is denoted by Δ. Let Γ be the graph formed by the shorter edges of these four triangles. The graph Γ is a star with the center p^O and four leafs $p^{ss'}, p^{st'}, p^{ts'}, p^{tt'}$ and edge lengths are $1/2$. Then T-dual is again reduced to

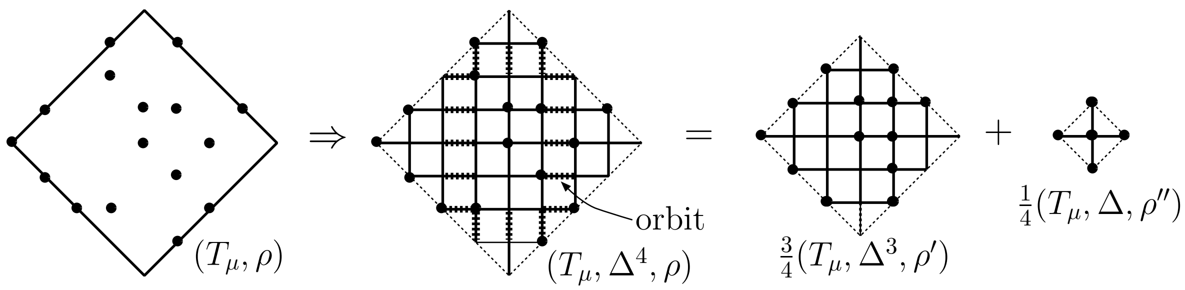

Figure 3. Subdividing and decomposing T_μ

the discrete location problem on Γ:

$$\text{(3.7)} \qquad \begin{aligned} &\text{Minimize } \frac{1}{2}\sum_{xy \in EG} c(xy)\mathrm{dist}_\Gamma(x,y) \\ &\text{subject to } \rho : VG \to V\Gamma, \\ &\qquad \rho(s) \in \{p^{ss'}, p^{st'}\},\ \rho(s') \in \{p^{ss'}, p^{ts'}\}, \\ &\qquad \rho(t) \in \{p^{ts'}, p^{tt'}\},\ \rho(t') \in \{p^{st'}, p^{tt'}\}. \end{aligned}$$

Indeed, for any (rational) potential $\rho : VG \to T_\mu$, the corresponding metric d^ρ defined by (2.10) can be represented as a convex combination of d^{ρ_i} $(i \in I)$ for potentials ρ_i satisfying the constraints of (3.7). We give an intuitive proof of this fact by using illustrations. We may assume that the image of ρ is rational. Then we can further subdivide T_μ into $1/k$-smaller squares and isosceles right triangles so that the image of ρ lies on the vertices of this subdivision as in Figure 3. This subdivision is denoted by Δ^k. We choose a set O of edges, called an *orbit*, of this subdivision by the following way. Take an arbitrary edge e of this subdivision, and set $O = \{e\}$. If there is a square having $e' \in O$ and $e'' \notin O$ as a parallel pair of edges, then set $O \leftarrow O \cup \{e''\}$. If there is a triangle having $e' \in O$ and $e'' \notin O$, then set $O \leftarrow O \cup \{e''\}$. Then all edges are partitioned into k orbits.

Contract all edges in O. Then we obtain $(k-1)/kT_\mu$. Expand it in factor $k/(k-1)$. From this, we obtain a feasible potential $\rho' : VG \to T_\mu$ whose the image lies on the vertices of the subdivision Δ^{k-1}. We construct one more potential. Contract all edges not in O. Then we obtain $1/kT_\mu$. Expand it in factor k. From this, we obtain a potential $\rho' : VG \to T_\mu$ whose the image lies on the vertices of Δ. Then one can easily see that

$$\text{(3.8)} \qquad d^\rho = \frac{k-1}{k} d^{\rho'} + \frac{1}{k} d^{\rho''}.$$

Repeat this process to ρ'. We obtain a desired convex combination.

Moreover, we can take an optimum ρ satisfying $(\rho(s), \rho(t)) = (\rho(s'), \rho(t')) = (p^{ss'}, p^{tt'})$ or $(\rho(s), \rho(t)) = (\rho(t'), \rho(s')) = (p^{st'}, p^{ts'})$. From this, we obtain Hu's max-biflow min-cut theorem [14] (without half-integrality of optimal flows):

$$\begin{aligned} &\text{the optimal multiflow value} \\ &= \min\{ss'\text{-}tt' \text{ mincut value}, \ st'\text{-}ts' \text{ mincut value}\}. \end{aligned} \tag{3.9}$$

§ 4. Geometry of T_μ and a general combinatorial min-max formula

In this section, we explain the constriction of Z in Theorem 1.2. The essential idea has already been described in the two-commodity example in Section 3.3. Namely, subdivide T_μ into squares and isosceles right triangles. Then we can take Z to be vertices of this subdivision. The following two propositions guarantee that this approach indeed works. The first one concerns the shape of 2-faces of T_μ.

Proposition 4.1. *Let F be a 2-face of T_μ. Then the metric space (F, l_∞) is isomorphic to the polygon Q in the l_∞-plane represented as*

$$Q = \left\{ (x_1, x_2) \in \mathbf{R}^2 \;\middle|\; \begin{array}{l} a_1 \leq x_1 \leq a_1', b \leq x_1 + x_2 \leq b', \\ a_2 \leq x_2 \leq a_2', c \leq x_1 - x_2 \leq c' \end{array} \right\} \tag{4.1}$$

for some $a_1, a_1', a_2, a_2', b, b', c, c' \in \mathbf{R}$. Moreover, the isometry is given by the projection $\mathbf{R}^S \to \mathbf{R}^{\{s,t\}}$ for some $s, t \in S$.

A polygon represented as (4.1) is exactly a convex polygon each of whose edges is parallel to one of the four vectors $(1,0), (0,1), (1,1), (1,-1)$. Recall that the l_∞-plane is the l_1-plane. By the map $(x_1, x_2) \mapsto ((x_1 + x_2)/2, (x_1 - x_2)/2)$, we again obtain a convex polygon in the l_1-plane each of whose edges is parallel to one of the four vectors $(1,0), (0,1), (1,1), (1,-1)$. If we draw the l_1/l_∞-coordinate on a 2-face F, then we observe that there are two types of edges of F: edges parallel to an l_1-axis and edges parallel to an l_∞-axis. Here an l_1-*axis* means a vector $(0,1)$ or $(1,0)$, and an l_∞-*axis* means a vector $(1,1)$ or $(1,-1)$ by the isometric projection to $(\mathbf{R}^2, l_\infty)$ in Proposition 4.1.

The second one says that if $\dim T_\mu \leq 2$, the metric space (T_μ, l_∞) is constructed by gluing such polygons *along the same type of edges*; see Figure 4 (a).

Proposition 4.2. *Suppose* $\dim T_\mu \leq 2$*. Let F, F' be 2-faces of T_μ sharing an edge e. The edge e is parallel to an l_1-axis on F if and only if e is parallel to an l_1-axis on F'.*

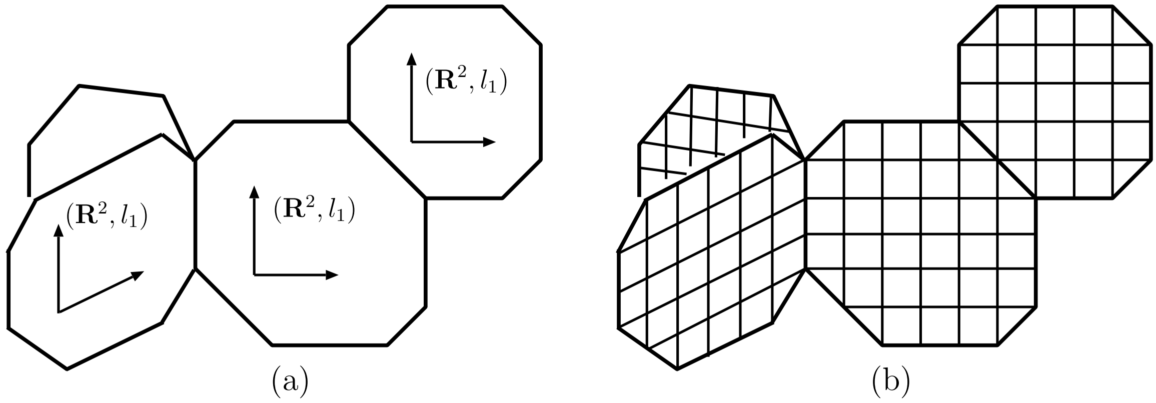

Figure 4. (a) gluing 2-faces and (b) an l_1-grid

In the following, we prove the first proposition (Proposition 4.1) by explaining a basic method to investigate T_μ, which will be often used in the subsequent arguments. For a point $p \in P_\mu$, we define an undirected graph $K(p)$ on $S = VK(p)$ by

$$st \in EK(p) \overset{\text{def}}{\Longleftrightarrow} p(s) + p(t) = \mu(s,t) \quad (s,t \in S). \tag{4.2}$$

Note that a loop appears at $s \in S$ exactly when $p(s) = 0$. The graph $K(p)$ expresses the information of facets of P_μ which contain p.

Take p^* in the relative interior of a face F. Suppose that $K(p^*)$ has m bipartite components with bipartitions $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}$. Then it is easy to see that the set of vectors $\{\chi_{A_i} - \chi_{B_i}\}_{i=1}^m$ is a basis of the vector space $\{p \in \mathbf{R}^S \mid p(s)+p(t) = 0 \ (st \in EK(p^*))\}$. Then every point p in F is uniquely represented as

$$p = p^* + \sum_{i=1}^{m} x_i(\chi_{A_i} - \chi_{B_i}) \tag{4.3}$$

for $(x_1, x_2, \ldots, x_m) \in \mathbf{R}^m$. Therefore we have the following.

Proposition 4.3 ([7]). *For $p \in T_\mu$, let $F(p)$ be the face containing p as its relative interior. Then we have*

$$\dim F(p) = \textit{the number of bipartite components of } K(p), \tag{4.4}$$

where loops are regarded as odd cycles.

In the expression (4.3), the map $p \mapsto (x_1, x_2 \ldots, x_m)$ is an injective isometry from (F, l_∞) to $(\mathbf{R}^m, l_\infty)$ since each $\chi_{A_i} - \chi_{B_i}$ is a 0-1 vector. From this fact, we easily obtain Proposition 4.1. Indeed, By substituting (4.3) with $k = 2$ to linear inequalities

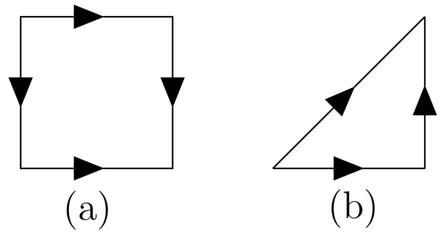

Figure 5. Orientation

$p(s) + p(t) \geq \mu(s,t)$ $(s, t \in S)$, we obtain the linear inequality representation (4.1). In particular, the isometry is given by the projection $\mathbf{R}^S \to \mathbf{R}^{\{s,t\}}$ for $s \in A_1 \cup B_1, t \in A_2 \cup B_2$.

§ 4.1. l_1-grids

Suppose that μ is a rational distance with $\dim T_\mu \leq 2$. By Propositions 4.1 and 4.2, there exists a polyhedral subdivision Δ of T_μ satisfying the following conditions:

(*1) 2-faces of Δ consist of squares and isosceles right triangles.

(*2) for each square $F \in \Delta$, its edges are parallel to l_1-axes of the 2-face containing F, and for each triangle $F \in \Delta$, its shorter edges are parallel to l_1-axes of the 2-face containing F.

We call a polyhedral subdivision Δ of T_μ with (*1-2) an l_1-*grid*; see Figure 4 (b). An edge of Δ is called an l_1-*edge* if it is not the longer edge of a triangle in Δ. If all l_1-edges of Δ have the same length α, then we call Δ α-*uniform*. A uniform l_1-grid always exists if μ is rational. The graph of l_1-edges realizes the l_∞-distance on T_μ as follows.

Proposition 4.4. *Let Δ be an l_1-grid of T_μ. For two vertices p, q in Δ, there is a path of length $\|p - q\|_\infty$ between p and q consisting of l_1-edges of Δ.*

The finite set Z in Theorem 1.2 can be taken to be the vertices of an l_1-grid satisfying a certain orientability condition. An l_1-grid Δ is *orientable* if edges of Δ are oriented so that

(o1) a parallel pair of edges of each square has the same direction, and

(o2) an acute angle of each triangle is a sink or a source.

See Figure 5. Let Δ be an orientable α-uniform l_1-grid Δ for a rational α. Let Γ be the graph of l_1-edges of Δ. Then we have the following.

Proposition 4.5. *There exists an optimal potential ρ in T-dual with $\rho(VG) \subseteq V\Gamma$ for any capacitated-graph (G, c) with $S \subseteq VG$*

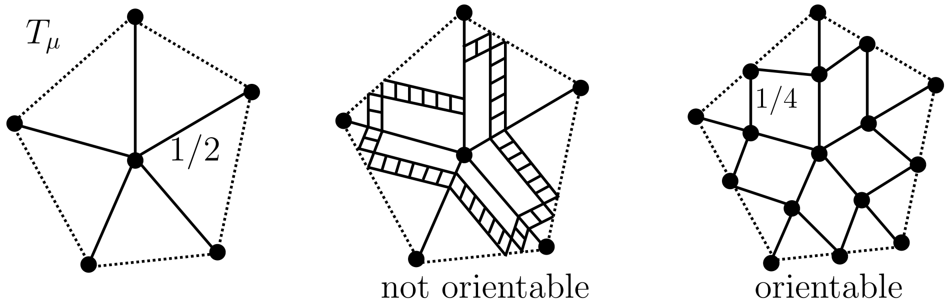

Figure 6. Tight span of the 0-1 distance of commodity graph C_5

Let $V_s = V\Gamma \cap T_{\mu,s}$ be the subset of vertices contained by $T_{\mu,s}$. Then T-dual is discretized into the following. This is a general combinatorial dual that yields previously known multiflow duality relations.

$$\begin{aligned}
(4.5) \qquad \text{Minimize} \quad & \alpha \sum_{xy \in EG} c(xy)\mathrm{dist}_\Gamma(\rho(x), \rho(y)) \\
\text{subject to} \quad & \rho : VG \to V\Gamma, \\
& \rho(s) \in V_s \quad (s \in S).
\end{aligned}$$

The proof of Proposition 4.5 is essentially the same as the proof of two-commodity example in Section 3.3. We do not repeat it here. Instead, we explain why the orientability condition is required. Consider the 0-1 distance μ of commodity graph C_5 (five cycle). Then T_μ is a pentagon obtained by gluing five isosceles right triangles along their right angle. Therefore T_μ has the 1/2-uniform l_1-grid Δ consisting of these five triangles. Then Δ is not orientable. If we apply the method used in two-commodity example, then an orbit of the subdivided l_1-grid Δ^k goes around the original l_1-grid Δ *twice* and the image of ρ'' does not lies on the vertices of Δ. In this case, Δ^2 is a desired orientable 1/4-uniform l_1-grid. This method making Δ orientable by subdivision is called an *orbit splitting* [11]; it is a slight modification of the original definition given in [23]. In particular, if an l_1-grid exists, then an orientable l_1-grid always exists.

§ 4.2. Half-integrality, lattice, and the folder decomposition

If there is an orientable $1/k$-uniform l_1-grid for a positve integer k, then we can take an $1/k$-integral optimal solution in (2.1) by Proposition 4.4 and the correspondence $\rho \mapsto d^\rho$ in (2.10).

Theorem 4.6 ([11]). *If μ is an integral distance, then there exists an orientable 1/4-uniform l_1-grid, and thus there exists a 1/4-integral optimal solution in LP-dual (2.1) of μ-problem.*

To show the existence of the 1/4-uniform l_1-grid is not difficult. This immediately follows from the fact that the polyhedron P_μ is half-integral if μ is integral. To show the orientability of this 1/4-uniform l_1-grid is not so easy. Here we give a sketch of this fact for the case of metric μ. We state it in a shaper form. An integral metric μ is called a *cyclically even* if $\mu(x,y)+\mu(y,z)+\mu(z,x)$ is even for $x,y,z \in S$. Clearly 2μ is cyclically even for every integral metric μ.

Theorem 4.7. *If μ is a cyclically even metric, then there exists an orientable 1/2-uniform l_1-grid.*

This theorem is essentially due to Karzanov [23]. His approach is graph-theoretical. Here we describe a different approach using a lattice (a discrete subgroup) in $\mathbf{R}^S$. Let L be a lattice in $\mathbf{R}^S$ defined by

$$L = \{p \in \mathbf{R}^S \mid p(s)+p(t) \in \mathbf{Z}\ (s,t \in S)\}. \tag{4.6}$$

This lattice is known as the *weight lattice of type B* in the representation theory of semisimple Lie algebras. Note that $L \subseteq (1/2)\mathbf{Z}^S$ and all extreme points of T_μ lie on L. By cyclically evenness, a simple calculation shows

$$\mu_s - \mu_t \in 2L \quad (s,t \in S), \tag{4.7}$$

where μ_s is s-th row vector of μ defined by (2.11). So we can define the affine lattice A_μ by

$$A_\mu = \mu_s + 2L. \tag{4.8}$$

Consider the graph Γ_o of A_μ by connecting pairs of points in A_μ having unit l_∞-distance. Namely, $V\Gamma_o = A_\mu$ and $pq \in E\Gamma_o$ if $\|p-q\|_\infty = 1$, or equivalently, $p-q \in \{1,-1\}^S$. Let Γ be the subgraph of Γ_o induced by $T_\mu \cap A_\mu$. For 2-face F of T_μ, the projection of $F \cap A_\mu$ to $\mathbf{R}^2$ coincides with the intersection of a polygon and a translation of the lattice

$$\{(x_1,x_2) \in \mathbf{Z}^2 \mid x_1+x_2 \in 2\mathbf{Z}\}. \tag{4.9}$$

This immediately follows from (4.3). Figure 7 illustrates A_μ and L with a 2-face. If all extreme points of T_μ belong to A_μ, then Γ coincides with the graph of the integral uniform l_1-grid of T_μ. However there may exist an extreme point of T_μ not in A_μ as indicated by the arrow in Figure 7.

Delete edges $E\Gamma$ from T_μ and consider (the closure of) the connected components. Then the connected components are classified into the following:

(1) a square formed by a 4-cycle in Γ lying on some 2-face.

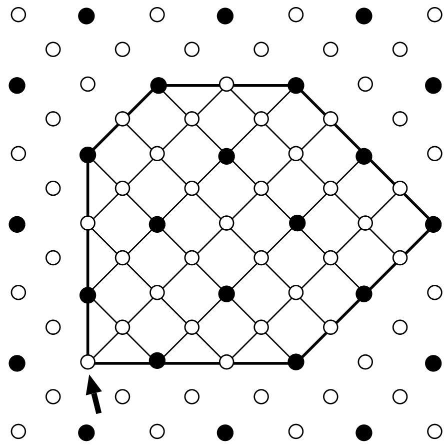

Figure 7. A_μ ($\bullet$), L ($\circ$), and 2-face

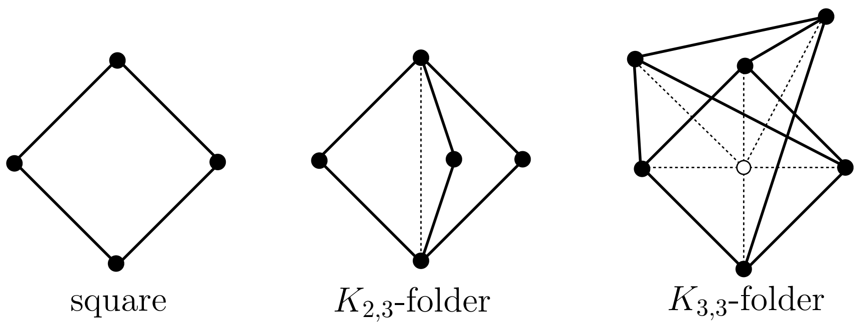

Figure 8. Folders

(2) a set containing a part of an edge of T_μ parallel to an l_∞-axis of some 2-face, which is obtained by gluing $m(\geq 3)$ isosceles right triangles along the common longer edge and is the interior of a subgraph $K_{2,m}$ of Γ.

(3) a set containing an extreme point p not belonging to A_μ, which is obtained by taking join of p and a complete bipartite subgraph $K_{n,m}$ $(n, m \geq 3)$ of Γ.

They are called *folders*. A folder of (1), (2), and (3) is called a *square*, a $K_{2,m}$*-folder*, and a $K_{n,m}$*-folder*, respectively. See Figure 8 for three types of folders. The decomposition of T_μ into these folders is called the *folder decomposition*.

By subdividing each folder and orienting its graph as in Figure 9, we obtain the 1/2-uniform l_1-grid Δ with orientation. Therefore the 1/2-uniform l_1-grid Δ is orientable. Thus we have Theorem 4.7. In particular, the vertices of this 1/2-uniform l_1-grid Δ is given explicitly by the intersection of T_μ and the lattice L. Then the discrete version of

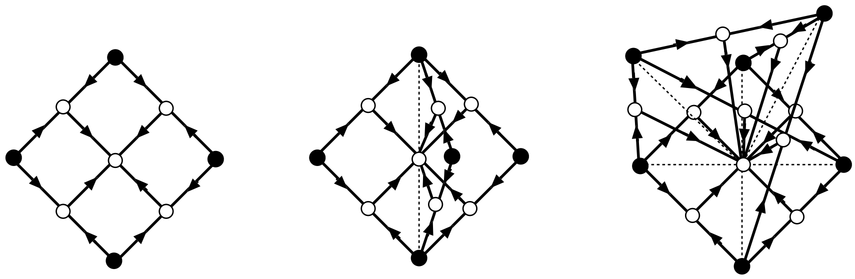

Figure 9. Subdividing and orienting folders

T-dual (4.5) is also represented as a *discrete convex optimization on the lattice* L:

$$\begin{aligned} \text{(4.10)} \qquad \text{Minimize} \quad & \sum_{xy \in EG} c(xy) \|\rho(x) - \rho(y)\|_\infty \\ \text{subject to} \quad & \rho : VG \to T_\mu \cap L, \\ & \rho(s) \in T_{\mu,s} \cap L \ (s \in S). \end{aligned}$$

This problem resembles an *L-convex function minimization* in Murota's theory of discrete convex analysis [27]. It would be interesting to explore such an analogy and develop *discrete convexity theory for multiflows.*

§ 4.3. Unbounded fractionality

In Section 4.1, we saw that 2-dimensionality of T_μ brings a combinatorial duality relation to μ-problem. On the other hand, we cannot expect such a combinatorial duality relation for the case $\dim T_\mu \geq 3$. Here we explain this fact. Recall that LP-dual (2.1) to μ-problem is a linear optimization of the polyhedron

$$\text{(4.11)} \qquad \mathcal{P}_{\mu,V} = \{d : \text{metric on } V(\supseteq S) \mid d(s,t) \geq \mu(s,t) \ (s,t \in S)\} + \mathbf{R}_+^{V \times V}.$$

Theorem 4.8 ([11]). *For an integral distance μ on S, if $\dim T_\mu \geq 3$, then there is no positive integer k such that $\mathcal{P}_{\mu,V}$ is $1/k$-integral for every V with $V \supseteq S$.*

This follows from the fact that there exists an infinite increasing series of finite subsets $P_1 \subseteq P_2 \subseteq \cdots$ in $(\mathbf{R}^3, l_\infty)$ such that the corresponding metrics are extreme in the metric cones. The standard TDI argument shows the following corollary.

Corollary 4.9. *If $\dim T_\mu \geq 3$, then there is no positive integer k such that μ-problem has a $1/k$-integral optimal multiflow for every integer-capacitated graph (G,c) with $S \subseteq VG$.*

§ 5. An application to metric packing

In this section, we describe an application of the folder decomposition of 2-dimensional tight spans, introduced in Section 4.2, to metric packing problems. The basic idea of such an approach is due to Chepoi [3]. Extending his approach, we solve Karzanov's conjecture concerning metric packing problems for the case where the commodity graph is vertex-disjoint sum of two triangles.

Let us introduce *metric packing problems*. Let G, H be undirected graphs with $VH \subseteq VG$. H is called a *commodity graph*. We assume that G is connected. A set of metrics $\{\mu_i\}_{i=1}^m$ on VG is called an *H-packing* if it satisfies

$$\sum_{i=1}^{k} \mu_i(x,y) \leq \mathrm{dist}_G(x,y) \quad (x,y \in VG), \tag{5.1}$$

$$\sum_{i=1}^{k} \mu_i(s,t) = \mathrm{dist}_G(s,t) \quad (st \in EH). \tag{5.2}$$

The existence of an H-packing by special combinatorial metrics is of central interest. The most simplest metric is a cut metric. A metric μ on V is called a *cut metric* if there is $X \subseteq V$ such that

$$\mu(x,y) = \begin{cases} 1 & \text{if } x \neq y, \#(\{x,y\} \cap X) = 1, \\ 0 & \text{otherwise} \end{cases} \qquad (x,y \in V). \tag{5.3}$$

A classical theorem in the network flow theory, often called the *max-potential min-work theorem*, says:

Theorem 5.1 ([30]). *If $H = K_2$, then there exists an H-packing by cut metrics.*

This is a *polar* theorem of Ford-Fulkerson's max-flow min-cut theorem. In fact, a metric packing problem is known to be polar to a multiflow feasibility problem; see [31, Section 70.12].

Karzanov extended Theorem 5.1 to the following multiterminal version, which generalizes Seymour's two-commodity cut packing theorem [32], and also strengthens Papernov's characterization of commodity graphs with the property that the cut condition is sufficient for the multiflow feasibility [29].

Theorem 5.2 ([18]). *If G is bipartite and H is K_4, C_5 or the union of two stars, then there exists an H-packing by cut metrics.*

If H is none of those graphs in this theorem, then an H-packing by cut metrics does not exist in general. However, by using some class of metrics beyond cut metrics, one can expect further combinatorial metric packing results. To describe it, we need some notation. For a graph Γ, a metric μ on V is called a *Γ-metric* if there is a map $\phi : V \to V\Gamma$ such that

$$\mu(x,y) = \mathrm{dist}_{\Gamma}(\phi(x), \phi(y)) \quad (x, y \in V). \tag{5.4}$$

In particular, a cut metric is just a K_2-metric. For a set $\mathcal{G}$ of graphs, a *$\mathcal{G}$-metric* is a Γ-metric for some $\Gamma \in \mathcal{G}$. Extending Theorem 5.2, Karzanov showed the following.

Theorem 5.3 ([20]). *If G is bipartite and H is K_5 or the union of a star and a triangle, then there exists an H-packing by $\{K_2, K_{2,3}\}$-metrics.*

It is known that if H has 3-matching, there is no *finite* set $\mathcal{G}$ of graphs with the property that every graph G with $VH \subseteq VG$ admits H-packing by proportions of $\mathcal{G}$-metrics [20, Section 3]. The graphs without 3-matching are classified into the following:

(1) K_4, C_5, and the union of two stars.

(2) K_5 and the union of a star and a triangle.

(3) the vertex-disjoint union of two triangles.

Theorems 5.2 and 5.3 solve the cases (1) and (2), respectively. For the case (3), Karzanov conjectured that there is an H-packing by $\{K_2, K_{2,3}, 1/2\Gamma_{3,3}\}$-metrics, where $\Gamma_{3,3}$ is the graph obtained by subdividing $K_{3,3}$ and connecting edges between all subdivided points and newly added one point [20, Section 3]. Namely, $\Gamma_{3,3}$ is the graph of the subdivision of the $K_{3,3}$-folder in Figure 9. Recently, [12] solved this conjecture in a stronger form.

Theorem 5.4 ([12]). *If G is bipartite and H is the vertex-disjoint union of two triangles, then there exists an H-packing by $\{K_2, K_{2,3}, K_{3,3}, \Gamma_{3,3}\}$-metrics.*

Note that a $K_{3,3}$-metric is a submetric of a $1/2\ \Gamma_{3,3}$-metric.

§ 5.1. Extremal graphs and geometry of T_μ

Here we sketch a proof of Theorem 5.4 given in [12]. For a metric μ on V, an *extremal graph* H of μ is a graph with $VH \subseteq V$ satisfying the following property:

(*) for any distinct $x, y \in V$, there exists $st \in EH$ such that

$$\mu(s,t) = \mu(s,x) + \mu(x,y) + \mu(y,t). \tag{5.5}$$

This means that every pair $x, y \in V$ is a part of a shortest path of some $s, t \in VH$. Metric packing problems in bipartite graphs with commodity graph H are reduced to a problem of decomposing cyclically even metrics having H as its extremal graph. Recall that an integral metric μ is called cyclically even if $\mu(x,y) + \mu(y,z) + \mu(z,x)$ is even for any x, y, z.

Lemma 5.5. *Let H be a graph. Let $\mathcal{G}$ be a finite set of graphs. If any cyclically even metric μ having H as an extremal graph is decomposed into an integral sum of $\mathcal{G}$-metrics, then every bipartite graph G with commodity graph H admits an H-packing by $\mathcal{G}$-metrics.*

For a proof of this fact, see [20, pp. 476–477]. Therefore, it suffices to consider the decomposition property of cyclically even metrics having H as its extremal graph. The following proposition connects extremal graph H and geometry of T_μ.

Proposition 5.6. *If an extremal graph H of a metric μ has no k-matching. then the dimension of T_μ is at most $k-1$.*

Proof. Suppose that $\dim T_\mu \geq k$. By Proposition 4.3, there is $p \in T_\mu$ such that $K(p)$ has k bipartite components. We show that each bipartite component has at least one edge of H. Take an edge xy from a bipartite component of $K(p)$. Then, by definition of $K(p)$, we have

$$p(x) + p(y) = \mu(x, y). \tag{5.6}$$

For some $s, t \in VH$, we have

$$\begin{aligned} \mu(s,t) \leq p(s) + p(t) &= p(s) - p(x) + p(x) + p(y) - p(y) + p(s) \\ &\leq \mu(s,x) + \mu(x,y) + \mu(y,t) = \mu(s,t), \end{aligned}$$

where we use Lemma 2.3 (4) in the second inequality. This implies $st \in EK(p)$, and similarly $sy, xt \in EK(p)$. $\square$

Therefore, if H has no 3-matching, then T_μ is 2-dimensional, and thus we can apply the folder decomposition of 2-dimensional tight spans, introduced in Section 4.2. Let Γ be the graph of $T_\mu \cap A_\mu$. By Lemma 2.3 and (a slight modification of) Proposition 4.4, μ is a submetric of dist_Γ. We decompose dist_Γ by using orbits as in Section 3.3. A pair of edges e, e' of Γ is called *projective* if there is a sequence of edges $e = e_1, e_2, \ldots, e_m = e'$ such that e_i and e_{i+1} are edges of some folder of type (2) or (3), or are parallel edges of a folder of type (1). The projectivity is an equivalence relation on $E\Gamma$. An equivalence class is called an *orbit*. Let $\mathcal{O}$ be the set of all orbits of Γ. For an orbit $o \in \mathcal{O}$, the *orbit graph* Γ^o is the graph obtained from Γ by contracting edges $E\Gamma \setminus o$ and deleting parallel edges appeared. This construction naturally gives a map $\phi^o : V\Gamma \to V\Gamma^o$ by defining $\phi^o(p)$ to be the contracted point. Then the following formula holds:

$$\mathrm{dist}_\Gamma(p,q) = \sum_{o \in \mathcal{O}} \mathrm{dist}_{\Gamma^o}(\phi^o(p), \phi^o(q)) \quad (p, q \in V\Gamma). \tag{5.7}$$

This is a special case of the decomposition of a modular graph by Bandelt [1] or the *canonical metric representation* of a bipartite graph by Lomonosov and Sebö [25]; also see [6, Section 20.1]. Therefore, it suffices to determine orbit graphs of Γ. By analyzing T_μ, one can show the following.

Lemma 5.7 ([12]). *If an extremal graph H of a cyclically even metric μ is the vertex-disjoint sum of two triangles, then an orbit graph of Γ is K_2, $K_{2,3}$, $K_{3,3}$, or an isometric subgraph of $\Gamma_{3,3}$.*

Thus we obtain Theorem 5.4.

§ 6. Concluding remarks

In this paper, we explained a unified approach to multiflow problems by using tight spans. We think that the potential of such a T_X-approach has not yet been fully exploited. Finally, we explain further related topics, future research directions, and open questions.

Minimum 0-extensions and minimizable graphs Here we explain a relationship between tight spans and minimum 0-extension problems discovered by Karzanov [22]. For an undirected graph G with nonnegative capacity $c \in \mathbf{R}_+^{EG}$ and an undirected graph Γ with $V\Gamma \subseteq V$, the *minimum 0-extension problem* is:

$$\begin{aligned} \text{(6.1)} \qquad \text{Minimize} \quad & \sum_{xy \in EG} c(xy)\mathrm{dist}_\Gamma(\rho(x), \rho(y)) \\ \text{subject to} \quad & \rho : VG \to V\Gamma, \\ & \rho(s) = s \ (s \in V\Gamma). \end{aligned}$$

This problem is NP-hard since it contains the 3-terminal cut problem for $\Gamma = K_3$. Karzanov considered the following relaxation problem:

$$\begin{aligned} \text{(6.2)} \qquad \text{Minimize} \quad & \sum_{xy \in EG} c(xy)d(x,y) \\ \text{subject to} \quad & d\text{: metric on } VG, \\ & d(s,t) = \mathrm{dist}_\Gamma(s,t) \ (s,t \in V\Gamma). \end{aligned}$$

Γ is said to be *minimizable* if (6.1) and (6.2) have the same optimal value for every capacitated graph (G, c) with $V\Gamma \subseteq VG$. Karzanov gave an elegant characterization of minimizable graphs as follows.

Theorem 6.1 ([22]). *Γ is minimizable if and only if Γ is bipartite, has no isometric cycles of length $k \geq 6$, and orientable.*

Here a graph Γ is *orientable* if Γ can be oriented so that the orientations of edges pq and rs in every 4-cycle (p, q, r, s) are opposite along the cycle as in Figure 5 (a). A bipartite graph without isometric cycles of length $k \geq 6$ is just a *hereditary modular graph* [2]. Orbits and related concepts that we used for l_1-grids were originally introduced for a class of modular graphs [22, 23].

A relation to our approach using l_1-grids is explained as follows. The relaxation (6.2) is the LP-dual to the μ-problem for $\mu = \mathrm{dist}_\Gamma$; the equality of the constraint of (2.1) is attained since $\mu = \mathrm{dist}_\Gamma$ is a metric. Then Γ necessarily coincides with the graph of an orientable l_1-grid of T_μ, and (4.5) coincides with (6.1). Thus, T_{dist_Γ} is obtained by filling l_1-space to each 4-cycle in Γ as in Figure 8 [22, 23]. Conversely, the graph Γ of an orientable l_1-grid of T_μ for a (possibly nonmetric) distance μ is necessarily minimizable. However we do not know a complete graph-theoretical characterization of the graph Γ *together with* $V_s (s \in S)$ arisen in (4.5). We leave it a future research topic.

Fractionality of optimal multiflows. In Section 4, we gave a general combinatorial min-max relation to μ-problems for a distance μ with $\dim T_\mu \leq 2$. However, it is not a *fully* combinatorial min-max relation, since it says nothing about the existence of integral, half-integral, quarter-integral, or $1/k$-integral optimal multiflows for a fixed positive integer k. As seen in Section 4.3, we cannot expect that μ-problems for a distance μ with $\dim T_\mu \geq 3$ have such a combinatorial min-max relation.

For 0-1 distance case, Karzanov conjectured:

Conjecture 6.2 ([21]). *If commodity graph H having no isolated vertices satisfies the following condition:*

(P) *for every intersecting triple of maximal stable sets A, B, C, we have $A \cap B = B \cap C = C \cap A$,*

then there exists a positive integer k such that the maximum multiflow problem with respect to H for any integer-capacitated graph (G, c) with $VH \subseteq VG$ has a $1/k$-integral optimal multiflow.

Karzanov [19] showed that the condition (P) is a necessary condition for the existence of such a positive integer k, and gave a combinatorial min-max relation in this case. In fact, (P) is equivalent to the condition of the 2-dimensionality of the tight span of the 0-1 distance corresponding to H. A detailed description of T_μ for a 0-1 distance with the property (P) is given by [11], and a combinatorial min-max relation from (4.5) coincides with Karzanov's one. Therefore, the following conjecture extending the previous one might be reasonable.

Conjecture 6.3. *For a distance μ on S, if* $\dim T_\mu \leq 2$, *then there exists a* *positive integer k such that μ-problem for any integer-capacitated graph (G, c) with $S \subseteq VG$ has a $1/k$-integral optimal multiflow.*

Directed multiflows and tropical polytopes. It is natural to ask whether a *geometric dual similar to T-dual* exists for μ-problems on *digraphs*. The forthcoming paper [13] joint with S. Koichi answers this question; a part of this work appeared in his Ph.D. thesis [24]. For *not necessarily symmetric* distance $\mu : S \times S \to \mathbf{R}_+$, consider the following polyhedral sets:

$$P_\mu = \{(p, q) \in \mathbf{R}^{S\times S} \mid p(s) + q(t) \geq \mu(s, t) \ (s, t \in S)\}, \tag{6.3}$$

$$T_\mu = \text{the set of minimal elements of } (P_\mu \cap \mathbf{R}_+^{S\times S}). \tag{6.4}$$

Then T_μ plays the same role of tight spans. Furthermore, when we restrict μ-problems on *Eulerian digraphs*, the following subset $\mathcal{T}_\mu$ of T_μ

$$\mathcal{T}_\mu = \mathbf{R}_+^{S\times S} \cap \{\text{the set of minimal elements of } P_\mu\} \tag{6.5}$$

gives sharper duality relations including Frank's directed version of free multiflow theorem [9] and Ibaraki-Karzanov-Nagamochi's directed version of the multiflow locking theorem [15]. Interestingly, $\mathcal{T}_\mu$ coincides with the intersection of the nonnegative orthant and a *tropical polytope* introduced by Sturmfels and Develin [5]. Then the dual of μ-problem for an Eulerian digraph is reduced to a certain location problem on a tropical polytope.

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