Metric Packing for $K_3 + K_3$

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Notation

$G = (V, E)$: an undirected graph with nonnegative capacity $c : E \to \mathbf{R}_+$
S: the set of terminals $S \subseteq V$
$\mathcal{P}$: the set of paths in G whose ends belong to S.

Definition. $f : \mathcal{P} \to \mathbf{R}_+$ is a *multiflow* (w.r.t $(G, c; S)$) if

$$\sum_{P \in \mathcal{P} : e \in P} f(P) \leq c(e) \quad (e \in E).$$

Multiflow feasibility problem

$G = (V, E)$: an undirected graph with nonnegative capacity $c \in \mathbf{R}_+^E$
$H = (S, R)$: a demand graph $S \subseteq V$

Given a demand $q : R \to \mathbf{R}_+$, find a multiflow $f : \mathcal{P} \to \mathbf{R}_+$ such that

$$\sum\{f(P) \mid P \in \mathcal{P} : P \text{ is } st\text{-path}\} = q(st) \quad (st \in R).$$

Japanese Theorem (Onaga-Kakusho 71, Iri 71)

There exists a feasible multiflow if and only if

$$\langle c, d \rangle_E \geq \langle q, d \rangle_R \quad (\forall d\text{: metric on } V).$$

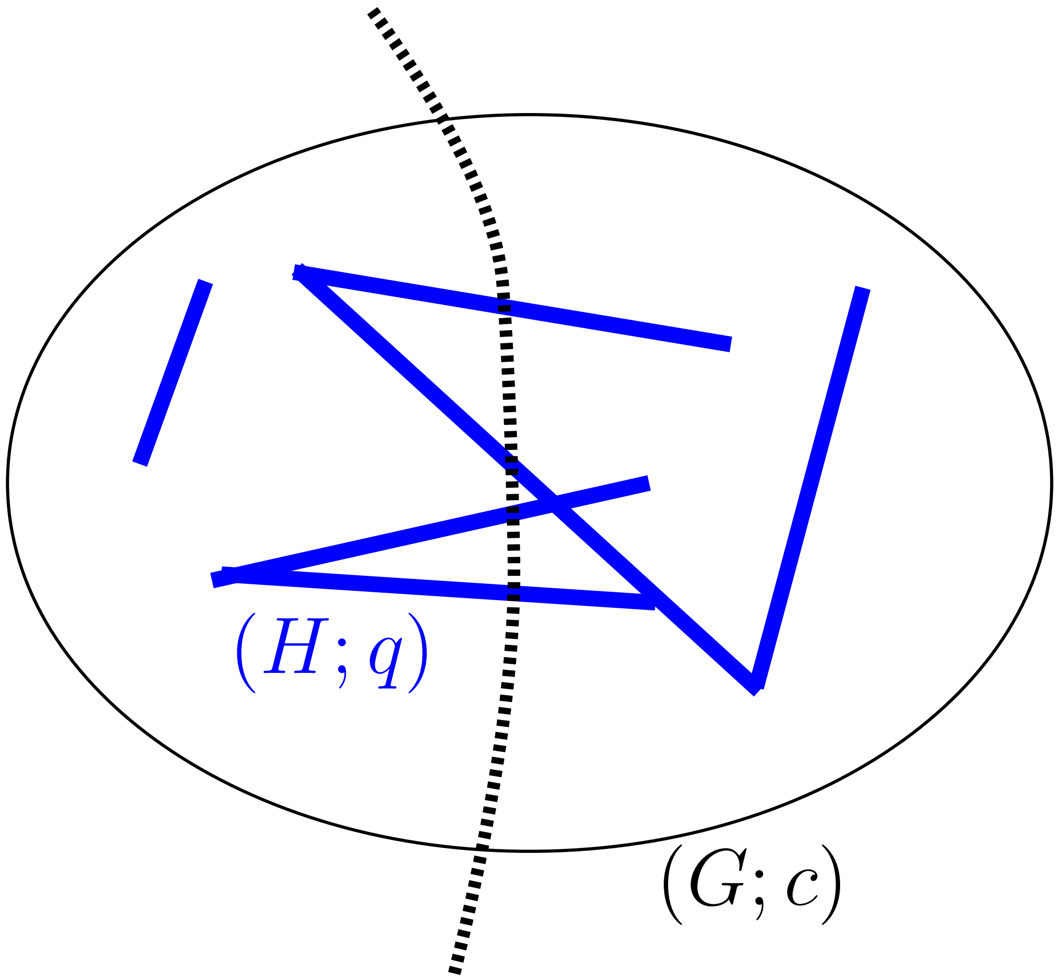
Cut condition:

$$\langle c, \delta_A \rangle_E \geq \langle q, \delta_A \rangle_R \quad (A \subseteq V)$$

Cut metric:

$\delta_A =$

	A	$V \setminus A$
A	0	1
$V \setminus A$	1	0



Cut condition: $\langle c, \delta_A \rangle_E \geq \langle q, \delta_A \rangle_R$ $(\forall A \subseteq V)$
Metric condition: $\langle c, d \rangle_E \geq \langle q, d \rangle_R$ ($\forall d$: metric).

When is the cut condition sufficient ?

Theorem (Papernov 76)
The cut condition is sufficient if and only if $H = K_4, C_5$ or the union of two star.

Theorem (Hu 63, Rothchild-Winston 66, Lomonosov 76, 85, Seymour 80)
If H is above and $G + H$ is Eulerian, then the cut condition implies an integer multiflow.

Polarity

Lemma (Seymour 79, Karzanov 84)
The cut condition is sufficient if and only if for any $l \in \mathbf{R}_+^E$ there are a family of cuts $\{\delta_{A_i}\}_i$ and its nonnegative weight $\{\lambda_i\}_i$ such that

$$\begin{aligned} \sum_i \lambda_i \delta_{A_i}(x,y) &\leq \text{dist}_{G,l}(x,y) \quad (x,y \in V), \\ \sum_i \lambda_i \delta_{A_i}(s,t) &= \text{dist}_{G,l}(s,t) \quad (st \in R) \end{aligned}$$

Such a $(\delta_{A_i}, \lambda_i)$ is called an fractional H-packing

Theorem (Seymour 80 for $H = K_2 + K_2$, Karzanov 85)
If H is above and G is bipartite, then there exists an integral H-packing by cut metrics.

Beyond the cut condition

Γ: undirected graph

Definition A metric d on V is called a Γ-metric if there is $\phi : V \to V\Gamma$ such that

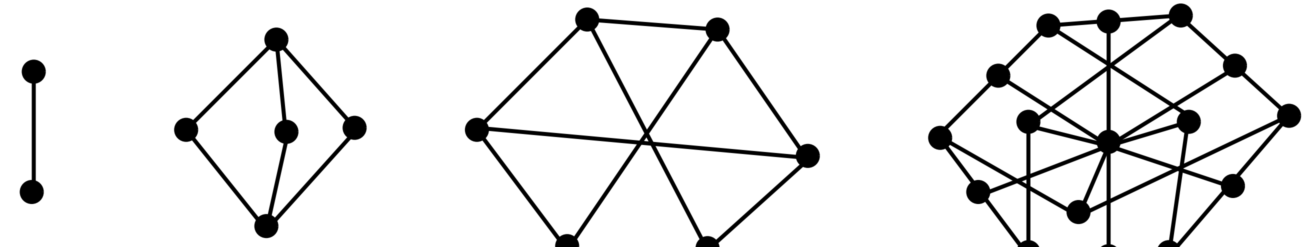
$$d(x, y) = \mathrm{dist}_{\Gamma}(\phi(x), \phi(y)) \quad (x, y \in V).$$

Remark: cut metric = K_2-metric.

Lemma: For a set $\mathcal{G}$ of graphs, $\mathcal{G}$-metric condition is sufficient if and only if for $l \in \mathbf{R}_+^E$ there are family of $\mathcal{G}$-metrics $\{d_i\}_i$ and its nonnegative weight $\{\lambda_i\}_i$ such that

$$\begin{aligned} \sum_i \lambda_i d_i(x, y) &\leq \mathrm{dist}_{G,l}(x, y) \quad (x, y \in V) \\ \sum_i \lambda_i d_i(s, t) &= \mathrm{dist}_{G,l}(s, t) \quad (st \in R) \end{aligned}$$

demand graph H	K_4, C_5, star + star	K_5, K_3+star	$K_3 + K_3$	other classes: H has 3-matching
multiflow for $G + H$:Eulerian	integer flow	integer flow (Karzanov 87)	$\exists k$, $1/k$-flow conjectured (Karzanov 90)	no fixed integer k, $1/k$-flow (Lomonosov 85)
feasibility condition	K_2 cut condition	K_2, $K_{2,3}$ (Karzanov 87)	K_2, $K_{2,3}$, $\Gamma_{3,3}$ (Karzanov 89)	infinite family of graphs (Karzanov 90)
H-packing for G: bipartite	integer packing	integer packing (Karzanov 90)	half-integer packing conjectured (Karzanov 90)	

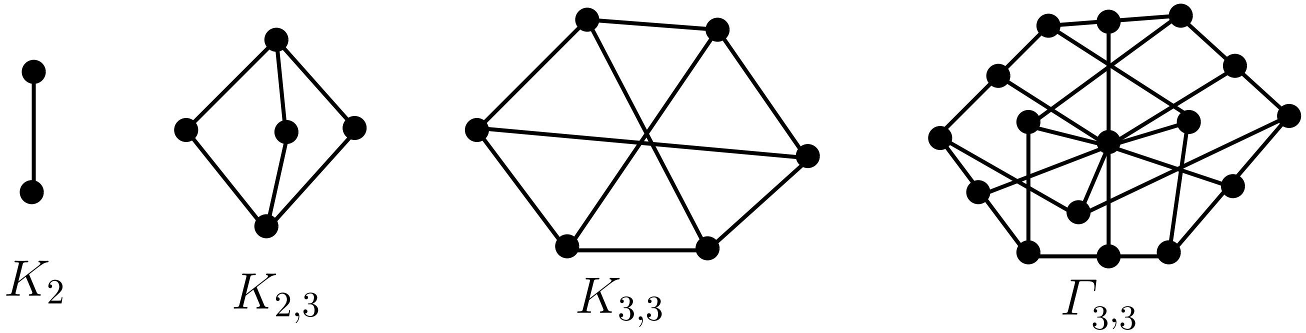


K_2 $K_{2,3}$ $K_{3,3}$ $\Gamma_{3,3}$

Main result

Theorem [H. 07]

If $H = K_3 + K_3$ and G is bipartite, then there is an integral H-packing by cut, $K_{2,3}$, $K_{3,3}$, and $\Gamma_{3,3}$-metrics



Definition:

A metric μ on V is *cyclically even* if $\mu(C)$ is even for every cycle C in K_V.

Definition:

An *extremal graph* $H = (S, R)$ of a metric μ is a graph on $S \subseteq V$ satisfying

$$\forall x, y \in V, \exists st \in R,\ \ \mu(s,t) = \mu(s,x) + \mu(x,y) + \mu(y,t).$$

Lemma (Karzanov 90)

An integral H-packing by $\mathcal{G}$-metrics for a bipartite graph
$\Leftarrow$ Decomposing cyclically even metric having H as an extremal graph into an integral sum of $\mathcal{G}$-metrics.

Definition:

μ: metric on V

$P_\mu = \{p \in \mathbf{R}^V \mid p(x) + p(y) \geq \mu(x, y) \ (x, y \in V)\}$

T_μ = the set of minimal elements of P_μ

T_μ:the tight span of μ (Isbell 64, Dress 84)

Embedding μ into (T_μ, l_∞) (Isbell 64, Dress 84)

μ_x: the x-th row vector of μ.

$\mu_x \in T_\mu$ and $\|\mu_x - \mu_y\|_\infty = \mu(x, y)$.

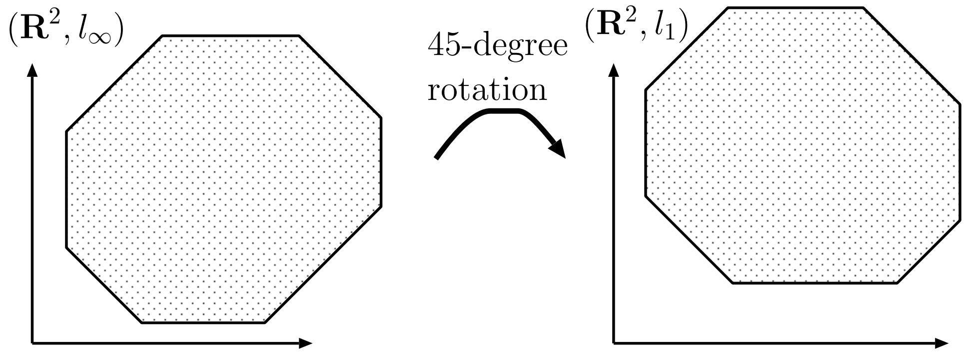
A key lemma(H. 07)

If an ex-graph H of metric μ has no k-matching, then

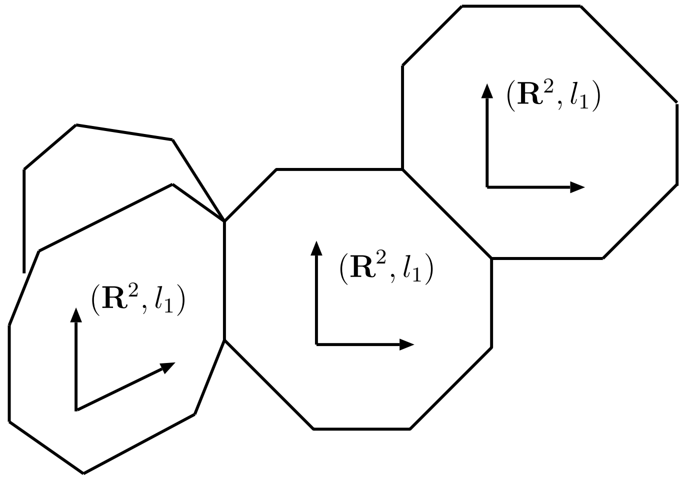
$$\dim T_\mu < k.$$

The shape of T_μ with $\dim T_\mu \leq 2$.

Lemma [H.07] 2-face of T_μ is isomorphic to



Lemma [H.07] 2-faces of T_μ are gluing *nicely*.



Affine lattice A_μ

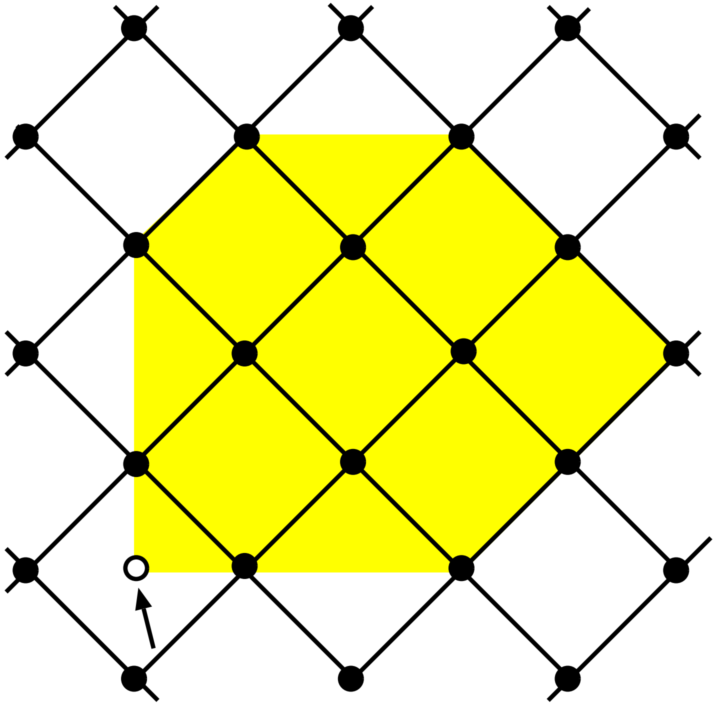
μ: a cyclically even metric on V

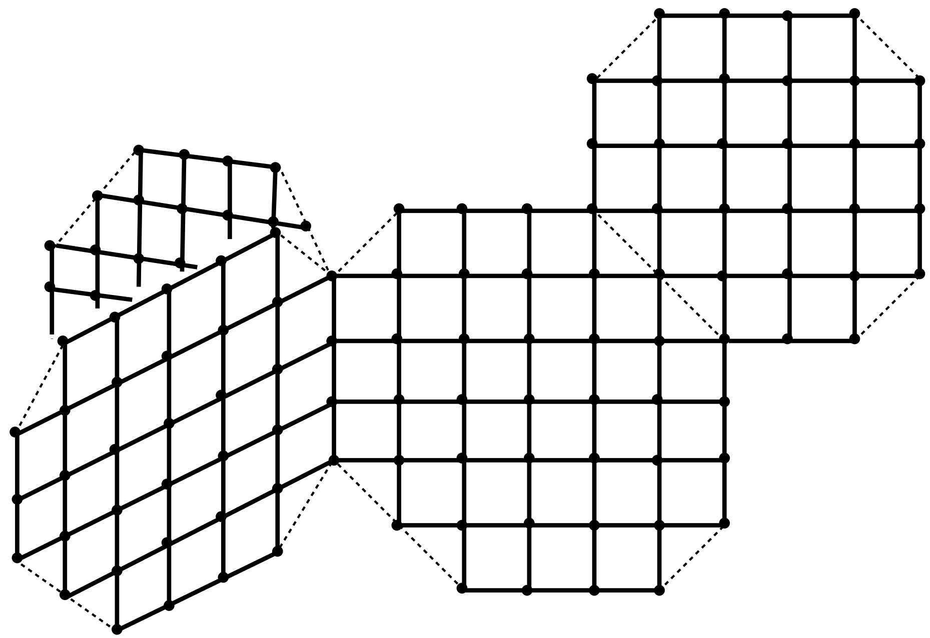
$L := \{p \in \mathbf{Z}^V \mid p(x) + p(y) \in 2\mathbf{Z} \quad (x, y \in V)\}$

$A_\mu := \mu_x + L$

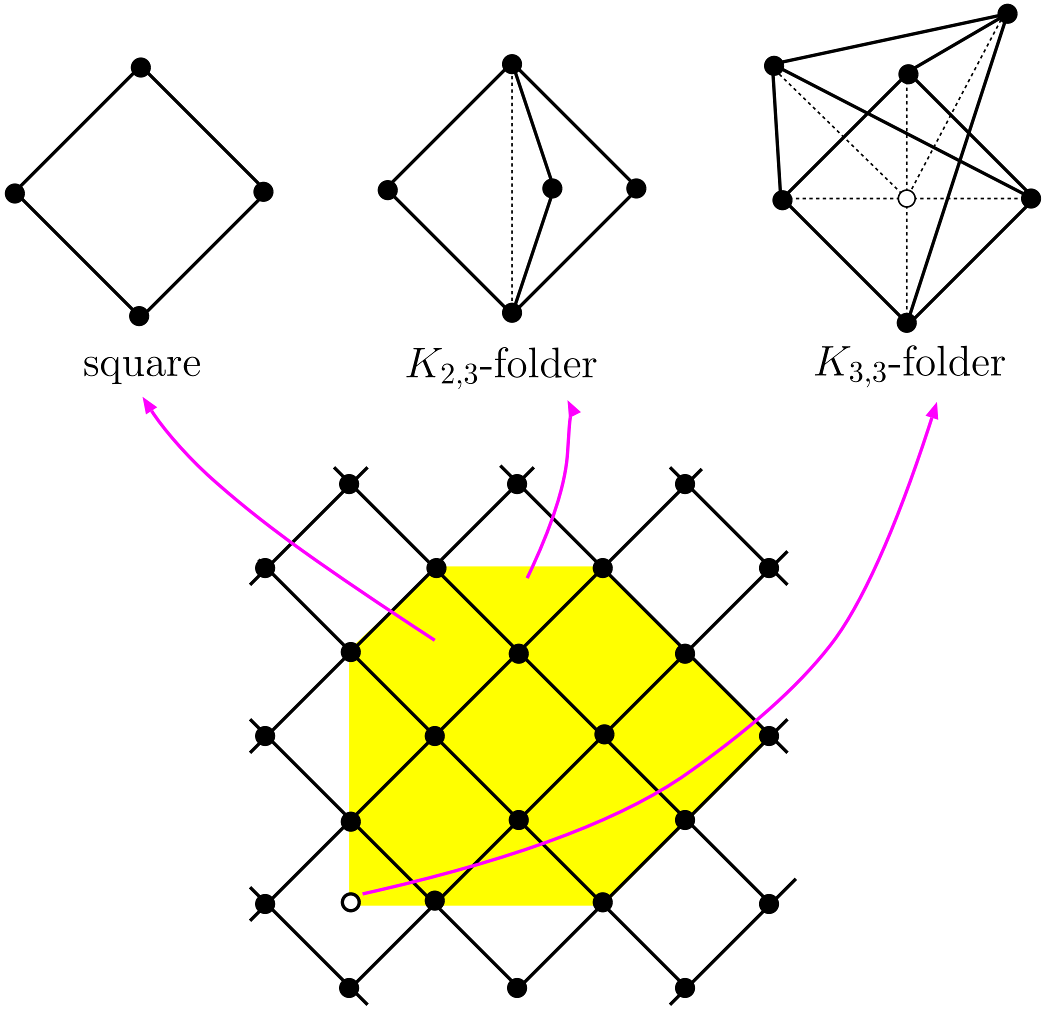
Γ_0: the graph of A_μ: $pq \in E\Gamma \Leftrightarrow p - q \in \{-1, 1\}^V$

Γ: the subgraph of Γ_0 induced by T_μ

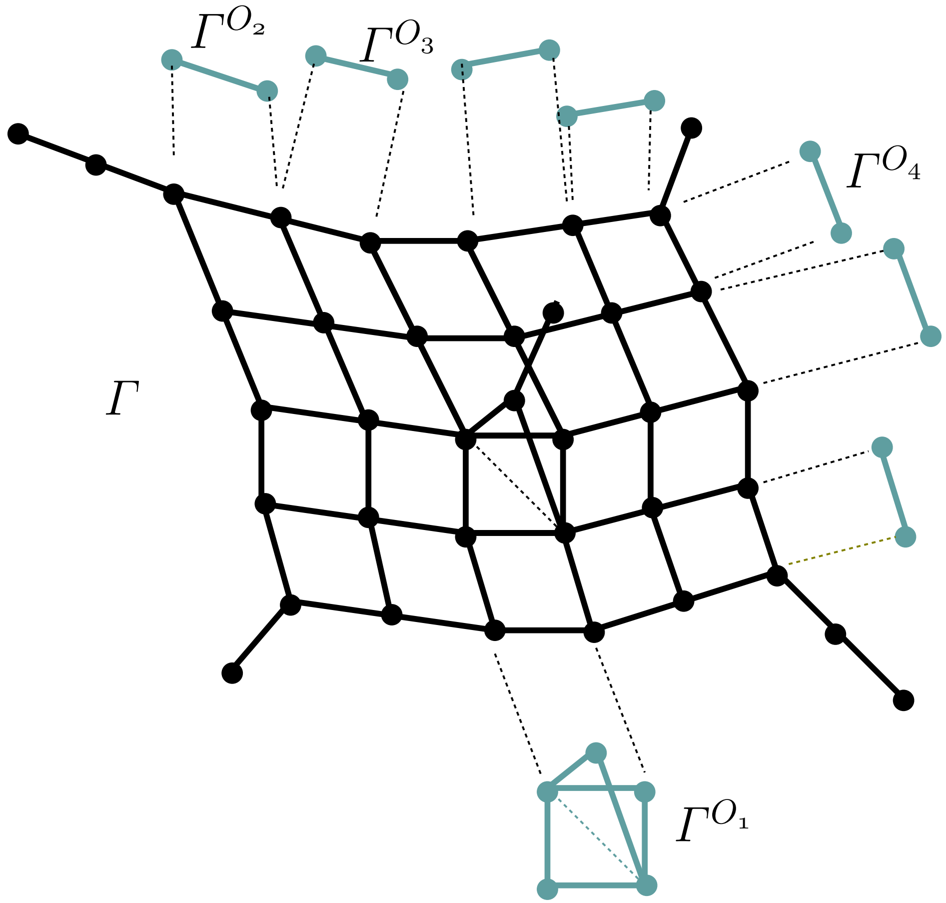




Proposition [H. 07] If an ex-graph H of a cyclically even metric μ has no 3-matching, the connected components of $T_\mu \setminus \Gamma$ are

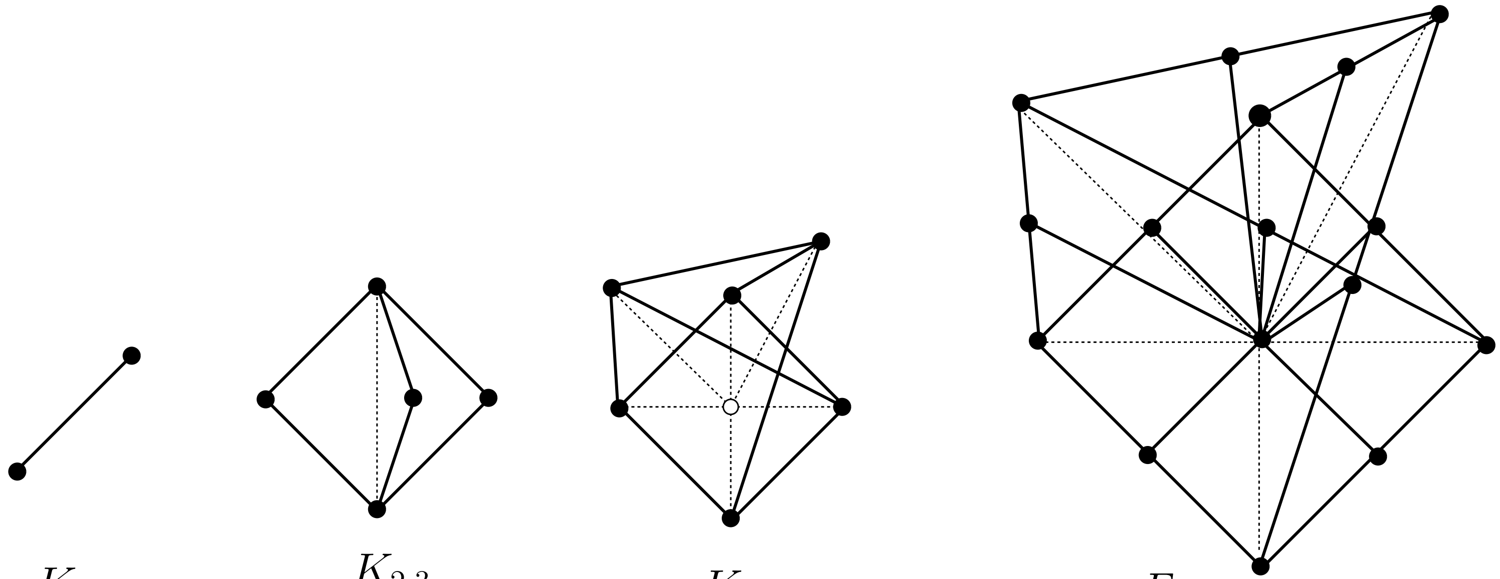


Orbit graph decomposition (Bandelt 85, Lomonosov & Sebö 93)



$$\mathrm{dist}_{\Gamma} = \mathrm{dist}_{\Gamma^{O_1}} + \mathrm{dist}_{\Gamma^{O_2}} + \cdots + \mathrm{dist}_{\Gamma^{O_m}}$$

Proposition [H. 07] If an ex-graph H of a cyclically even metric μ has no 3-matching, then an orbit graph of Γ is K_2, $K_{2,3}$, $K_{3,3}$, or an isometric subgraph of $\Gamma_{3,3}$.



K_2 $K_{2,3}$ $K_{3,3}$ $\Gamma_{3,3}$

Future works

- Feasible multiflows for demand graph $K_3 + K_3$ (in preparation)

- A unified understanding to planar multiflows and some variations:

  – planar multiflows with demand edges on k holes ($k = 1$: Okamura-Seymour 81, $k = 2$: Okamura 83, $k = 3, 4$: Karzanov 94,95)

  – graph having no K_5-minor (Seymour 81), signed graph having no odd K_5-minor (Geelen-Guenin 01)