Folder complexes and multiflow combinatorial dualities

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Abstract

In multiflow maximization problems, there are several combinatorial duality relations, such as Ford-Fulkerson's max-flow min-cut theorem for single commodity flows, Hu's max-biflow min-cut theorem for two-commodity flows, Lovász-Cherkassky duality theorem for free multiflows, and so on. In this paper, we provide a unified framework for such multiflow combinatorial dualities by using the notion of a *folder complex*, which is a certain 2-dimensional polyhedral complex introduced by Chepoi. We show that for a nonnegative weight μ on terminal set, μ-weighted maximum multiflow problem admits a combinatorial duality relation if and only if μ is represented by distances between certain subsets in a folder complex, and show that the corresponding combinatorial dual problem is a discrete location problem on the graph of the folder complex. This extends a result of Karzanov in the case of metric weights.

1 Introduction

Let G be an undirected graph with nonnegative edge capacity $c : EG \to \mathbf{R}_+$. Let $S \subseteq VG$ be a set of terminals. A path P in G is called an *S-path* if its ends belong to distinct terminals. A *multiflow* is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of S-paths and its nonnegative flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying capacity constraint $\sum_{P \in \mathcal{P}: e \in P} \lambda(P) \leq c(e)$ for $e \in EG$. For a nonnegative weight $\mu : \binom{S}{2} \to \mathbf{R}_+$ on the set of pairs of terminals, the *μ-weighted maximum multiflow problem* (*μ-problem* for short) is formulated as

$$(1.1) \qquad \begin{array}{ll} \text{Maximize} & \displaystyle\sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \\ \text{subject to} & (\mathcal{P}, \lambda) : \text{multiflow for } (G, c; S), \end{array}$$

where s_P, t_P denote the ends of P. The μ-problem (1.1) is a linear program. So there is a linearly programming duality. However, for a special weight μ, the μ-problem admits a *combinatorial duality relation*. For example, suppose that S is a 2-set $\{s, t\}$. Then the μ-problem is the maximum flow problem. The max-flow min-cut theorem, due to Ford-Fulkerson [8], says that the maximum flow value is equal to the minimum s-t cut value. Suppose that S is a four-set $\{s, t, s', t'\}$, and $\mu(s, t) = \mu(s', t') = 1$ and other weights are zero. Then the μ-problem is the two-commodity flow maximization. Hu [14] proved that the maximum multiflow value is equal to the minimum of ss'-tt' mincut value and st'-ts' mincut value. Suppose that μ is the all-one weight. Then Lovász [23] and Cherkassky [5] proved that the maximum multiflow value is equal to the half of the sum of t-$(S \setminus t)$ mincut values over $t \in S$.

A remarkable result by Karzanov [17] completely characterized *metric*-weights μ admitting such a combinatorial duality relation. Let us describe it. Here μ is called a *metric* if μ satisfies the triangle inequality. For an undirected graph Γ, the shortest path metric on the vertex set $V\Gamma$ is denoted by d_Γ. An *isometric* cycle C in Γ is a (simple) cycle such that $d_\Gamma(x, y) = d_C(x, y)$ holds for every pair x, y of vertices in C. An undirected graph Γ is called a *frame* if

(i) Γ is bipartite,

(ii) Γ has no isometric cycle of length greater than four, and

(iii) Γ is *orientable* in the sense that Γ has an orientation with the property that, for each 4-cycle C, nonadjacent edges have opposite directions with respect to a cyclic ordering of C; see the left of Figure 4.

Theorem 1.1 ([17]). *Let μ be a metric on a finite set S. Suppose that there exist a frame Γ and a map $\phi : S \to V\Gamma$ such that*

$$\mu(s, t) = d_\Gamma(\phi(s), \phi(t)) \quad (s, t \in S).$$

Then, for any capacitated graph (G, c) having S as terminal set, the maximum value of the μ-problem (1.1) *is equal to the minimum value of the following discrete location problem on Γ:*

$$\begin{array}{lll} (1.2) \qquad & \text{Minimize} & \displaystyle\sum_{xy \in EG} c(xy) d_\Gamma(\rho(x), \rho(y)) \\ & \text{subject to} & \rho : VG \to V\Gamma, \\ & & \rho(s) = \phi(s) \quad (s \in S). \end{array}$$

(Karzanov [17] actually presented this theorem in the dual form, i.e., the *minimum 0-extension problem* on a frame is solvable by its *metric relaxation*, which is the LP-dual to the metric-weighted maximum multiflow problem.) Moreover, if a rational metric μ cannot be represented by (a dilation of) a submetric of a frame, then there is no such a combinatorial duality relation [18]. From the point of the view in Theorem 1.1, Lovász-Cherkassky duality relation can be understood as follows. Let μ be the all-one distance on S. Then 2μ is represented as the distance between the leaves of a star consisting of $|S|$ edges. A star is clearly a frame. Then (1.2) coincides with the problem of finding a t-$(S \setminus t)$ mincut for each $t \in S$.

Unfortunately, this theorem cannot be applicable to μ-problems for *nonmetric* weights μ, such as two-commodity flows. Our previous paper [11] partially extended Theorems 1.1 to a nonmetric version. However this result needs tedious calculations for the tight span and its subdivision, which cause difficulty and inconvenience for a further combinatorial study of μ-problems (1.1). The main contribution (Theorem 2.1) of this paper is a natural extension of Theorem 1.1 for general weights, which provides a useful and flexible framework for multiflow combinatorial dualities. Also this result is an important step toward the complete classification of weights μ having finite fractionality [13].

To describe our result, we make use of the notion of a *folder complex* (an *F-complex* for short) introduced by Chepoi [3, Section 7]. Roughly speaking, an F-complex is a 2-dimensional polyhedral complex obtained by *filling a 2-dimensional cell into each 4-cycle of a frame*. In Section 2, we describe basic definitions and the main result. A key concept that we newly introduce here is the notion of a *normal set* in an F-complex. Our main result (Theorem 2.1) says that if weight μ can be realized by the distances between normal sets in an F-complex under l_1-metrization, then the μ-problem has a combinatorial dual problem similar to (1.2). We also give illustrative examples for multiflow combinatorial dualities by F-complexes, including Karzanov-Lomonosov duality relation for anticlique-bipartite commodity flows and seemingly new combinatorial duality relation for a *weighted version* of two-commodity flows.

Our main result immediately follows from two properties: one is the Helly property of normal sets, and the other one is a decomposition property of finite submetrics on an F-complex. This connection to Helly property sheds a new insight on the multiflow duality. To prove them, in Section 3, we reveal several intriguing properties of shortest paths connecting normal sets in an F-complex, which are previously known for shortest paths connecting vertices of a frame. These properties will also play key roles in the next paper [13]. In Section 4.2, we give a special optimality criterion to combinatorial dual problems, which extends one in the classical tree location problems. Also we verify that weight μ has such an F-complex realization if and only if μ has combinatorial dimension at most 2. This result has already been suggested by the previous paper [11].

Notation. Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of reals and nonnegative reals, respectively. The set of functions from a set V to $\mathbf{R}$ is denoted by $\mathbf{R}^V$. For $p, q \in \mathbf{R}^V$, $p \leq q$ means $p(x) \leq q(x)$ for all $x \in V$.

A function $\mu : S \times S \to \mathbf{R}$ is called a *distance* on S if $\mu(s,t) = \mu(t,s) \geq \mu(s,s) = 0$ for $s, t \in S$. We regard a nonnegative weight on terminals as a distance. A distance d on S is called a *metric* if d satisfies the triangle inequality $d(x,z) \leq d(x,y) + d(y,z)$ $(x, y, z \in S)$. For a metric d on S, the minimum distance between two subsets A and B is denoted by $d(A,B)$, i.e.,

$$d(A,B) = \inf_{x \in A, y \in B} d(x,y).$$

We simply denote $d(\{x\}, A)$ by $d(x,A)$. For a subset $A \subseteq S$ and a nonnegative real r, the *ball* $B(A,r)$ around A of radius r is defined by

$$B(A,r) = \{x \in S \mid d(A,x) \leq r\}.$$

For an undirected graph G, the vertex set and the edge set are denoted by VG and EG, respectively. An edge joining $x, y \in VG$ is simply denoted by xy or yx. Let d_G denote the shortest path metric on VG with some specified edge-length. If no edge length is specified, then d_G means the shortest path metric by unit edge-length. A subgraph G' of G is called *isometric* if $d_{G'}(x,y) = d_G(x,y)$ for $x, y \in VG'$.

Let $\mathcal{F} \subseteq 2^S$ be a set of subsets. We say "*$\mathcal{F}$ has the Helly property*" if every subset $\mathcal{F}' \subseteq \mathcal{F}$ having a pairwise nonempty intersection

$$A \cap B \neq \emptyset \quad (A, B \in \mathcal{F}')$$

has a nonempty intersection

$$\bigcap_{A \in \mathcal{F}'} A \neq \emptyset.$$

A *polyhedral complex* Δ is a set of polyhedra in some Euclidean space such that every face of $\sigma \in \Delta$ belongs to Δ, and the nonempty intersection of $\sigma, \sigma' \in \Delta$ is a face of both σ and σ'. Let $|\Delta|$ denote the underlying set $\bigcup_{\sigma \in \Delta} \sigma$. A member in Δ is called a *cell*. A k-dimensional cell is simply called a *k-cell*. A 1-cell is also called an *edge*. If an edge has ends p, q, then it is also denoted by pq. A k-dimensional polyhedral complex is simply called a *k-complex*. We always assume that *Δ is finite*.

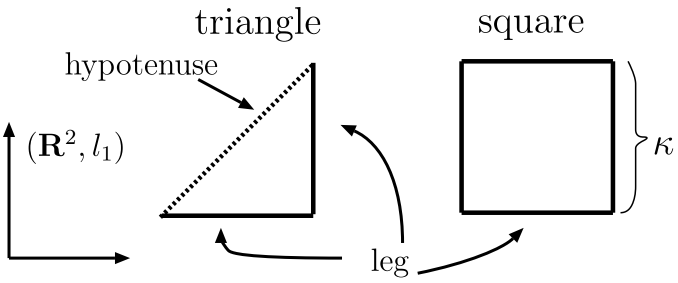

Figure 1: A square and a triangle equipped with the l_1-distance

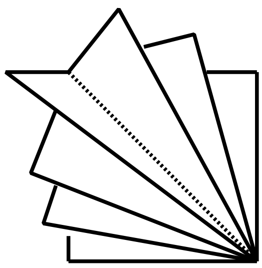

Figure 2: A folder

2 Definitions, results, and examples

Let $\varDelta$ be a (finite) 2-complex satisfying the following property:

(2.1) (i) For some positive rational κ, each maximal cell σ has an injective continuous map $\varphi_\sigma : \sigma \to \mathbf{R}^2$ such that the image $\varphi_\sigma(\sigma)$ is

(a) the convex hull of $\{(0,0),(\kappa,0),(0,\kappa),(\kappa,\kappa)\}$,

(b) the convex hull of $\{(0,0),(\kappa,0),(\kappa,\kappa)\}$, or

(c) the segment $[(0,0),(\kappa,0)]$, and

the image of a face of σ is a face of $\varphi_\sigma(\sigma)$.

(ii) If $\sigma \cap \sigma' \neq \emptyset$ for two maximal cells σ, σ', then $\varphi_{\sigma'} \circ \varphi_\sigma^{-1}$ is an isometry from $\varphi_\sigma(\sigma \cap \sigma')$ to $\varphi_{\sigma'}(\sigma \cap \sigma')$ in the Euclidean distance.

By condition (ii), if 2-cells σ, σ' share a common edge e, then $\varphi_\sigma(e) = [(0,0),(\kappa,\kappa)]$ implies $\varphi_{\sigma'}(e) = [(0,0),(\kappa,\kappa)]$. An edge e of $\varDelta$ is called a *hypotenuse* if $\varphi_\sigma(e) = [(0,0),(\kappa,\kappa)]$ for a 2-cell σ containing e. The other edges are called *legs*. In particular, a maximal 1-cell is a leg. κ is called the *leg-length*. A 2-cell σ is called a *square* if its image by φ_σ is (a), and is called a *triangle* if its image by φ_σ is (b). Namely $\varDelta$ is obtained by gluing squares and isosceles right triangles along same type of edges. A *folder* of $\varDelta$ is a square or the union of all triangles sharing one common hypotenuse; see Figure 2.

A more combinatorial and abstract approach is often useful. We can regard $\varDelta$ as a pair of a simple graph $\varGamma_0$ and a subset $\mathcal{C}$ of its chordless cycles such that

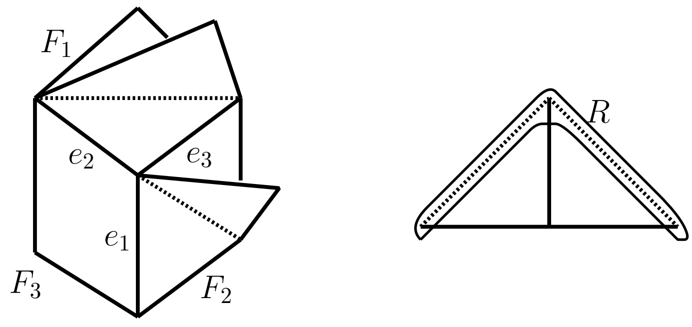

Figure 3: (a) violation of the flag condition, and (b) violation of the local convexity

(2.2) (i) the edge set of Γ_0 is partitioned into two types of edges, called *legs* and *hypotenuses*,

(ii) each member of $\mathcal{C}$ is

(a) a 3-cycle (a *triangle*) consisting of one hypotenuse and two legs, or

(b) a 4-cycle (a *square*) consisting of four legs,

and every hypotenuse belongs to some triangle, and

(iii) any two members in $\mathcal{C}$ intersect at one edge or one vertex, or do not intersect.

Indeed, we can take Γ_0 as the 1-skeleton graph of Δ, and take $\mathcal{C}$ as boundary cycles of 2-cells. Conversely, from $(\Gamma_0; \mathcal{C})$ with property (2.2) and a positive rational κ, we can construct a 2-complex Δ with property (2.1).

We can endow $|\Delta|$ with the l_p-metric by the following way. For each cell σ and a path P in σ, the *l_p-length* of P is defined by the l_p-length of $\varphi_{\sigma'}(P)$ in $(\mathbf{R}^2, l_p)$ for a maximal cell σ' containing σ (well-defined by (2.1) (ii)). Then we can define the *l_p-length* of a path P in $|\Delta|$ by the sum of the l_p-length of $P \cap \sigma^\circ$ over all cells σ, where σ° is the relative interior of σ. Thus we can define the metric d_{Δ,l_p} on $|\Delta|$ by defining $d_{\Delta,l_p}(p,q)$ to be the infimum of the lengths of all paths connecting p,q in $|\Delta|$. In this paper, we are mainly interested in the l_1-metrization d_{Δ,l_1}, which is simply denoted by d_Δ.

A simply-connected 2-complex Δ with property (2.1) is called a *folder complex* (an *F-complex* for short) [3, Section 7] if

(2.3) (i) the intersection of any two folders does not contain incident legs, and

(ii) there are no three folders F_i $(i = 1,2,3)$ and three distinct legs e_i $(i = 1,2,3)$ sharing a common vertex such that e_i belongs to F_j exactly when $i \neq j$.

In fact, this is equivalent to the condition that $|\Delta|$ is a CAT(0) space under the l_2-metrization [3, Theorem 7.1]. We particularly call (ii) the *flag condition*; see Figure 3 (a). We remark that (i) can be replaced by a stronger condition: (i') the intersection of any two folders is a leg, a vertex, or empty (by the property (3.6) (i) in Section 3.1).

Next we introduce certain subsets of $|\Delta|$. A subset R of $|\Delta|$ is called *normal* if it satisfies the following property:

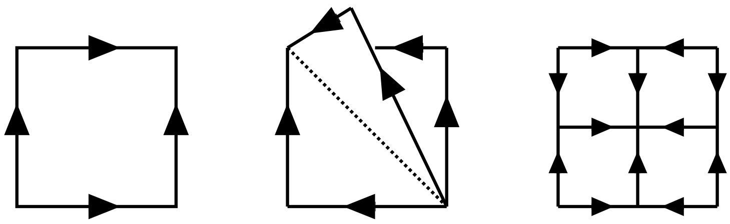

Figure 4: Orientation

(2.4) (i) R coincides with $|\mathcal{K}|$ for a connected subcomplex $\mathcal{K}$ of $\varDelta$ with the property that if $\mathcal{K}$ contains a leg e, then every cell containing e belongs to $\mathcal{K}$.

(ii) there are no two triangles σ and σ' sharing a leg and a right angle such that $(\sigma \cup \sigma') \cap R$ coincides with the union of hypotenuses of σ and σ'.

We particularly call (ii) the *local convexity condition*; see Figure 3 (b). Note that $\mathcal{K}$ may have a maximal face consisting of a hypotenuse. Let $\varGamma = \varGamma^{\varDelta}$ be the graph consisting of all legs of $\varDelta$, called the *leg-graph* of $\varDelta$. Equivalently, $\varGamma$ is the graph obtained by deleting all hypotenuses from the 1-skeleton graph of $\varDelta$. The edge-length of $\varGamma$ is defined by κ uniformly. One can easily see that any pair of vertices p, q in $\varDelta$ is joined by a path of length $d_{\varDelta}(p, q)$ consisting of legs. This means

$$d_{\varGamma}(p, q) = d_{\varDelta}(p, q) \quad (p, q \in V\varGamma).$$

An F-complex $\varDelta$ is said to be *orientable* if the leg-graph $\varGamma$ has an orientation such that

(2.5) (i) in each square, its diagonal edges have same direction, more precisely, orientations of two nonincident edges are opposite each other in a cyclic ordering of the corresponding 4-cycle, and

(ii) in each folder consisting of triangles, the hypotenuse joins a source and a sink.

See Figure 4. In fact, the leg-graph of an orientable F-complex is a frame; see Theorem 3.2 in Section 3.1.

For a rational distance μ on set S, a pair $(\varDelta; \{R_s\}_{s \in S})$ of an F-complex $\varDelta$ and a set $\{R_s\}_{s \in S}$ of normal sets is called an *F-complex realization* of μ if it satisfies

$$\mu(s, t) = d_{\varDelta}(R_s, R_t) \quad (s, t \in S).$$

Namely, μ is realized by the l_1-distances between normal sets R_s.

Consider the μ-problem (1.1) for a capacitated graph (G, c) with $S \subseteq VG$, and consider the following continuous and discrete location problems on $\varDelta$:

(2.6)
$$\begin{aligned} \text{Minimize} \quad & \sum_{xy \in EG} c(xy) d_{\varDelta}(\rho(x), \rho(y)) \\ \text{subject to} \quad & \rho : VG \to |\varDelta|, \\ & \rho(s) \in R_s \quad (s \in S). \end{aligned}$$

(2.7)
$$\begin{aligned} \text{Minimize} \quad & \sum_{xy \in EG} c(xy) d_{\varGamma}(\rho(x), \rho(y)) \\ \text{subject to} \quad & \rho : VG \to V\varGamma, \\ & \rho(s) \in R_s \cap V\varGamma \quad (s \in S). \end{aligned}$$

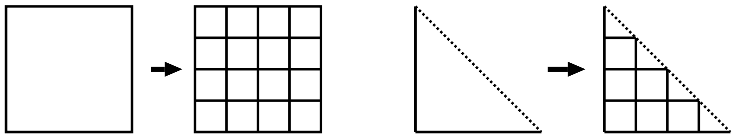

Figure 5: 4-subdivision

Now we are ready to describe our main result.

Theorem 2.1. *Let μ be a rational distance on S. Suppose that μ has an F-complex realization $(\varDelta; \{R_s\}_{s \in S})$. Then, for any capacitated graph (G, c) with $S \subseteq VG$, the following holds:*

(1) *The maximum value of the μ-problem* (1.1) *is equal to the minimum value of the continuous location problem* (2.6)*.*

(2) *In addition, if $\varDelta$ is orientable, then the maximum value of the μ-problem* (1.1) *is equal to the minimum value of the discrete location problem* (2.7)*.*

An F-complex $\varDelta$ may not be orientable. By the subdivision operation, we can always obtain an orientable F-complex as follows. For a positive integer $k \geq 2$, the *k-subdivision* $\varDelta^k$ is obtained from $\varDelta$ by subdividing each square into k^2 squares, subdividing each triangle into $k(k-1)/2$ squares and k triangles, and subdividing each maximal 1-cell(edge) into k edges as in Figure 5. The leg-length is set to be κ/k. It can be checked that:

(2.8) (i) The k-subdivision $\varDelta^k$ is an F-complex for any k.

(ii) The 2-subdivision $\varDelta^2$ is always orientable.

One can verify (i) from the fact that (2.3) is a local condition at each point, and verify (ii) from the right of Figure 4; orient it so that each vertex in $\varDelta$ is a sink, and the midpoint of each folder in $\varDelta$ is a source. Next we describe illustrative examples of combinatorial duality relations for μ-problems based on Theorem 2.1.

Two-commodity flows. Consider the case where terminal set S is partitioned into four subsets S, T, S', T' and distance μ on S is given as $\mu(s,t) = \mu(s',t') = 1$ for $(s,t,s',t') \in S \times T \times S' \times T'$ and zero for others. Then the μ-problem (1.1) is the multiterminal two-commodity flow maximization problem. We can realize μ by the distance on an F-complex as follow. Consider four triangles, glue them around a common right angle, and define the leg-length κ to be 1/2, as in Figure 6 (a). The resulting 2-complex is clearly an orientable F-complex. Moreover $\varDelta$ realizes μ as the distance between four hypotenuses, which are normal. Theorem 2.1 (ii) gives a combinatorial duality relation for the multiterminal two-commodity flow maximization [4]. In the case $|S| = |T| = |S'| = |T'| = 1$, one can see that an optimum is always attained by a map with image belonging to one of diagonal pairs in the boundary vertices. From this, one obtain Hu's max-biflow min-cut formula [14].

Weighted two-commodity flows. Also we can consider a *weighted* version of two-commodity flows. For relatively prime integers p, q, define μ as $(\mu(s,t), \mu(s',t')) = (p,q)$ for $(s,t,s',t') \in S \times T \times S' \times T'$ and zero for others. Then consider an F-complex $\varDelta$ as in Figure 6 (b), which is a subdivision of a rectangle of edge-length ratio $(p:q)$.

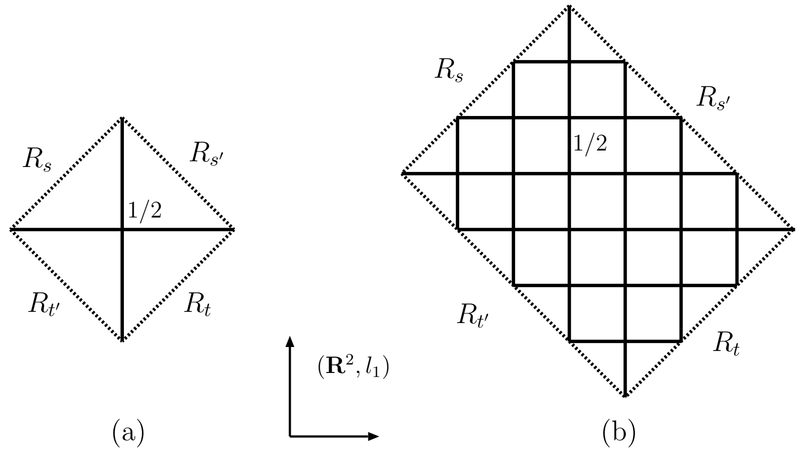

Figure 6: Two-commodity F-complexes

Let $\kappa = 1/2$. Clearly Δ is a F-complex. Again μ is realized by the distance of the four edges of the rectangle, which are normal. Also one can see that Δ is orientable. Then Theorem 2.1 (ii) yields a combinatorial duality relation, which seems new in the literature. Interestingly, in the case $|S| = |T| = |S'| = |T'| = 1$, the max-biflow min-cut formula again holds for every p, q (personal communication with A. Sebö, 2010).

Anticlique-bipartite commodity flows. A 0-1 distance μ can be identified with a *commodity graph* H on S by $st \in EH \Leftrightarrow \mu(s,t) = 1$. We may assume that H has no isolated vertices. The corresponding μ-problem for (G, c) is the maximization of the total sum of flow-values of multiflows connecting pairs of terminals specified by edges of H. For a multiflow f, the total flow-value with respect to H is denoted by $\|f\|_H$. A bipartition (X, Y) of vertex set VG is called a *cut*, and its *cut capacity* $c(X, Y)$ is defined to be the sum of the capacity of edges with exactly one end belonging to X. A *multicut* $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ $(k \geq 2)$ with respect to H is a partition of VG with the property that ends of each edge of H belong to distinct parts of $\mathcal{X}$, and its capacity $c(\mathcal{X})$ is the sum of the capacity of edges whose ends belong to distinct parts of $\mathcal{X}$. For a multiflow f and a multicut $\mathcal{X}$ (w.r.t. H), the following obvious weak duality relation holds:

$$\|f\|_H \leq c(\mathcal{X}). \tag{2.9}$$

In general, the duality gap is strict (except single- and two-commodity flows). Consider the following *relaxation* of multicuts. A *semi-multicut* $\mathcal{X} = \{X_1, X_2, \ldots, X_k\}$ $(k \geq 2)$ with respect to H is a *set of disjoint subsets* of VG with the property that ends of each edge of H belong to distinct sets in $\mathcal{X}$, and its capacity $c(\mathcal{X})$ is defined by

$$c(\mathcal{X}) = \frac{1}{2} \sum_{i=1}^{k} c(X_i, VG \setminus X_i).$$

Again the weak duality (2.9) holds for any multiflow and any semi-multicut, and the duality gap is still strict. However, the class of commodity graphs H admitting strong duality (for semi-multicuts) is not so narrow; recall Lovász-Cherkassky theorem for the case where H is a complete graph. Karzanov and Lomonosov [20] characterized such a class of commodity graphs H as follows.

Theorem 2.2 ([20]). *Suppose that the set $\mathcal{A}$ of maximal stable sets of commodity graph H has the following properties:*

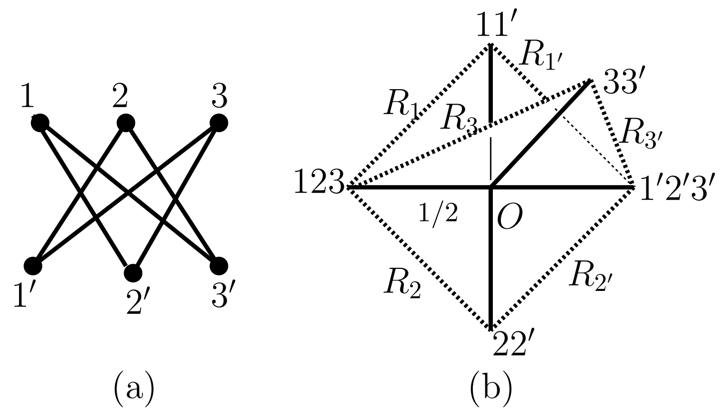

Figure 7: (a) an anticlique bipartite commodity graph and (b) its F-complex realization

(1) *For every triple $A, B, C \in \mathcal{A}$ of maximal stable sets, at least one of $A \cap B$, $B \cap C$, and $C \cap A$ is empty.*

(2) *The intersection graph of $\mathcal{A}$ is bipartite.*

Then, for any capacitated graph (G, c) having H as commodity graph, the following holds:

$$\max\{\|f\|_H \mid \text{multiflow } f\} = \min\{c(\mathcal{X}) \mid \text{semi-multicut } \mathcal{X}\}.$$

See [22], and also see [9] for a polymatroidal approach to this result. Note that the condition (1) is implied by (2). A commodity graph with condition (1-2) is called *anticlique-bipartite*. Here we give an interpretation of this results in terms of an F-complex. Let us construct an F-complex realization for 0-1 distance μ corresponding to a commodity graph H with property (1). Let $\Omega_{\mathcal{A}}$ be the intersection graph of $\mathcal{A}$. From $\Omega_{\mathcal{A}}$, we can construct an abstract 2-complex of (2.2). All edges of $\Omega_{\mathcal{A}}$ are supposed be hypotenuses. Add a new vertex O, and add a leg OA for each $A \in \mathcal{A}$. Let Γ_0 be the resulting graph. 2-cells $\mathcal{C}$ consists of 3-cycles of one hypotenuse AB and two legs OA, OB over all intersecting pairs $A, B \in \mathcal{A}$. Define the leg-length κ to be 1/2. Let Δ be the resulting 2-complex, which is the join of O and $\Omega_{\mathcal{A}}$. By the property (1), the girth of $\Omega_{\mathcal{A}}$ is at least four. This fact implies that Δ is an F-complex; the violation of the flag condition at O implies the existence of a chordless cycle in $\Omega_{\mathcal{A}}$ of length at most three. For $s \in S$, define a normal set R_s by

$$R_s = \begin{cases} \text{hypotenuse } AB & \text{if } s \text{ belongs to exactly two } A, B \in \mathcal{A}, \\ \text{vertex } A & \text{if } s \text{ belongs to exactly one } A \in \mathcal{A}. \end{cases}$$

Note that every terminal s belongs to at most two maximal stable sets by (1) and the assumption that H has no isolated vertices. Then $(\Delta; \{R_s\}_{s\in S})$ is an F-complex realization. Indeed, $d_\Delta(R_s, R_t) = 0 \Leftrightarrow R_s \cap R_t \neq \emptyset \Leftrightarrow R_s \cap R_t = \{A\}$ for $A \in \mathcal{A}$ with $s, t \in A \Leftrightarrow \mu(s, t) = 0$, and $R_s \cap R_t = \emptyset$ implies that $d_\Delta(R_s, R_t) = 1$. Furthermore Δ is orientable if and only if the condition (2) is fulfilled. Consider the discrete location problem (2.7) for $(\Delta; \{R_s\}_{s\in S})$. For a map $\rho : VG \to V\Gamma$ feasible to (2.7), the set $\mathcal{X} := \{\rho^{-1}(A) \mid A \in \mathcal{A}\}$ is a semi-multicut, and $\sum_{xy\in EG} c(xy) d_\Delta(\rho(x), \rho(y))$ coincides with $c(\mathcal{X})$. Thus Theorem 2.1 implies Karzanov-Lomonosov duality relation above.

Figure 7 illustrates (a) an anticlique-bipartite commodity graph H of $K_{3,3}$ minus one perfect matching, and (b) its F-complex realization Δ. In this case, Δ is obtained by gluing six triangles of leg-length 1/2, and each R_s is a hypotenuse.

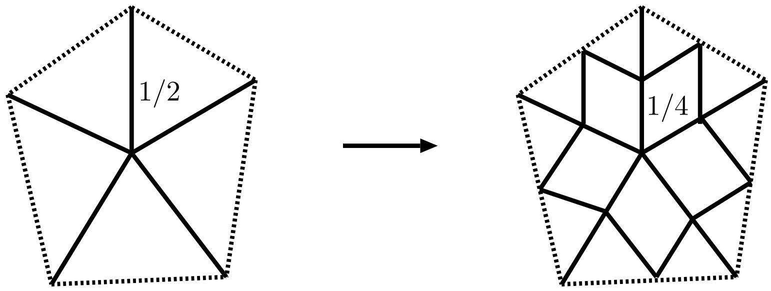

Figure 8: F-complexes for $H = C_5$

Also for nonorientable case, i.e., H violates (2), we obtain a combinatorial duality relation by subdividing Δ into Δ^2, which coincides with the one given in [16]. Figure 8 illustrates a nonorientable F-complex corresponding to the commodity graph of five-cycle, and its 2-subdivision. See the next paper [13] for further examples.

3 Folder complexes and normal sets

Our main Theorem 2.1 is a consequence of the following two properties of an F-complex Δ; see the paragraph below.

(3.1) (i) For any collection of normal sets $N_1, N_2, \ldots, N_k$ and nonnegative rationals $r_1, r_2, \ldots, r_k$, the collection of balls $\{B(N_i, r_i)\}_{i=1}^k$ has the Helly property.

(ii) Suppose that Δ is orientable. For a map $\rho : X \to |\Delta|$ on a finite set X, define metric d^ρ on X by

$$d^\rho(x, y) = d_\Delta(\rho(x), \rho(y)) \quad (x, y \in X).$$

Then there exist maps $\rho_i : X \to V\Gamma$ $(i \in I)$ such that d^ρ is a convex combination of d^{ρ_i} and, for any normal set R, $\rho(x) \in R$ implies $\rho_i(x) \in R$ $(i \in I)$.

(i) is an extension of [3, Theorem 7.8] in the case where each N_i is a singleton set; see Section 4.2. The goal of this section is prove (3.1). In Section 3.1, we introduce some of basic notation. In Section 3.2, we study shortest paths connecting normal sets. By using them, in Section 3.3 we prove (3.1) (i). In Section 3.4, we show a certain decomposition property of shortest paths, and prove (3.1) (ii).

Proof of Theorem 2.1. Here we show Theorem 2.1 by assuming (3.1). The second statement (2) of Theorem 2.1 immediately follows from the first (1) and (3.1) (ii). So we show the first statement. Let μ be a rational distance on a finite set S. For a capacitated graph (G, c) with terminal set S, consider the μ-problem (1.1). As is well-known [22], its LP-dual problem is given by

(3.2)
$$\begin{aligned} \text{Minimize} \quad & \sum_{xy \in EG} c(xy) m(x, y) \\ \text{subject to} \quad & m : \text{metric on } VG, \\ & m(s, t) \geq \mu(s, t) \quad (s, t \in S). \end{aligned}$$

Let $(\Delta; \{R_s\}_{s\in S})$ be an F-complex realization of μ. For a map ρ feasible to (2.6), define metric d^ρ on VG as in (3.1) (ii). Then condition $\rho(s) \in R_s$ implies

$$d^\rho(s,t) = d_\Delta(\rho(s), \rho(t)) \geq d_\Delta(R_s, R_t) = \mu(s,t),$$

and thus d^ρ is feasible. Hence it suffices to show:

(3.3) For every rational metric m feasible to (3.2), there exists a map $\rho : VG \to |\Delta|$ such that $d^\rho \leq m$ and $\rho(s) \in R_s$ for $s \in S$.

The following argument is based on the idea used in [1, p.421]. Let m be a feasible rational metric. Let $VG = \{x_1, x_2, \ldots, x_n\}$. Define a map $\rho : VG \to |\Delta|$ recursively by $\rho(x_k)$ being an arbitrary point in

$$\bigcap_{s\in S} B(R_s, m(s,x_k)) \cap \bigcap_{i=1}^{k-1} B(\rho(x_i), m(x_i,x_k)) \quad (k = 1,2,\ldots,n). \tag{3.4}$$

We show that the set (3.4) is nonempty for all k, which implies that ρ is well-defined and satisfies $\rho(s) \in R_s$ and $d^\rho \leq m$. Indeed, $\rho(s) \in B(R_s, m(s,s)) = R_s$, and $\rho(x_j) \in B(\rho(x_i), m(x_i,x_j))$ $(i < j)$ implies $d_\Delta(\rho(x_i), \rho(x_j)) \leq m(x_i,x_j)$.

We use the induction on k. Suppose that (3.4) is nonempty for $i < k$. By rationality, we can take $\rho(x_i)$ as a vertex of l-subdivision Δ^l for large l. Then each singleton set $\{\rho(x_i)\}$ is normal in Δ^l. Of course, R_s is also normal in Δ^l. By the Helly property (3.1) (i), it suffices to show that each pairwise intersection is nonempty. Note that two balls $B(R,r)$ and $B(R',r')$ intersect if and only if $d_\Delta(R,R') \leq r + r'$. For $s,t \in S$, the nonemptiness of $B(R_s, m(s,x_k)) \cap B(R_t, m(t,x_k))$ follows from $m(s,x_k) + m(x_k,t) \geq m(s,t) \geq \mu(s,t) = d_\Delta(R_s,R_t)$. For $s \in S$ and $i < k$, the nonemptiness of $B(R_s, m(s,x_k)) \cap B(\rho(x_i), m(x_i,x_k))$ follows from $m(s,x_k) + m(x_k,x_i) \geq m(s,x_i) \geq d_\Delta(R_s, \rho(x_i))$, where the last inequality follows from $\rho(x_i) \in B(R_s, m(s,x_i))$ for $i < k$ (by induction). For $1 \leq i < j < k$, the nonemptiness of $B(\rho(x_i), m(x_i,x_k)) \cap B(\rho(x_j), m(x_j,x_k))$ follows from $m(x_i,x_k) + m(x_k,x_j) \geq m(x_i,x_j) \geq d_\Delta(\rho(x_i), \rho(x_j))$. Thus we are done.

3.1 Preliminaries

Let Γ be an undirected graph. The *interval* $I_\Gamma(x,y)$ between $x,y \in V\Gamma$ is defined to be the set $\{z \in VG \mid d_\Gamma(x,y) = d_\Gamma(x,z) + d_\Gamma(z,y)\}$ (for some specified edge length). An undirected graph Γ (with unit edge length) is called *modular* if for each triple x,y,z of vertices the intersection $I_\Gamma(x,y) \cap I_\Gamma(y,z) \cap I_\Gamma(z,y)$ is nonempty. An element in $I_\Gamma(x,y) \cap I_\Gamma(y,z) \cap I_\Gamma(z,y)$ is called a *median* of x,y,z. A modular graph Γ satisfies the so-called *quadrangle condition*:

(3.5) For any four vertices x,u,v,y with $d_\Gamma(x,u) = d_\Gamma(x,v) = 1$ and $d_\Gamma(u,y) = d_\Gamma(v,y) = d_\Gamma(x,y) - 1$, there exists a common neighbor w of u and v such that $d_\Gamma(w,y) = d_\Gamma(x,y) - 2$.

Indeed, we can take w as a median of u,v,y.

A graph is called *hereditary modular* if every isometric subgraph is modular. Bandelt [2] gave an elegant characterization of hereditary modular graphs as follows.

Theorem 3.1 ([2]). *A graph is hereditary modular if and only if it is bipartite and has no isometric cycles of length greater than four.*

Therefore a frame is just an orientable hereditary modular graph. Chepoi [3] established a relation between hereditary modular graphs and F-complexes. Here a folder consisting of a single triangle is called a *triangle-folder*.

Theorem 3.2 ([3, Theorem 7.1])**.** *For a simply-connected* 2*-complex* Δ *satisfying* (2.1) *without triangle-folders, the following conditions are equivalent:*

(i) *Δ is an F-complex.*

(ii) *its leg-graph Γ is a hereditary modular graph without $K_{3,3}$ and $K_{3,3}^-$ as induced subgraphs.*

In addition, (i) $\Rightarrow$ (ii) *holds for any simply-connected* 2*-complex* Δ *satisfying* (2.1) *(possibly including triangle-folders).*

Here $K_{3,3}^-$ is the graph obtained from $K_{3,3}$ by deleting one edge. Both $K_{3,3}$ and $K_{3,3}^-$ are nonorientable [18]; one can verify it by hand. So the leg-graph of an orientable F-complex is a frame (with or without triangle-folders). We can partially reconstruct an F-complex Δ from its leg-graph Γ based on the following property:

(3.6) (i) every 4-cycle of the leg-graph Γ belongs to a (unique) folder.

(ii) every 3-cycle of the 1-skeleton graph of Δ, consisting of two legs and one hypotenuse, is the boundary of a triangle.

This property follows from the argument in [3, Section 7] (with or without triangle-folders). We sketch to prove it for completeness.

Sketch of the proof of (3.6). Take a (simple) cycle C in the leg-graph Γ. Since Δ is simply-connected, C is a boundary of some disk D in $|\Delta|$; see [3, Section 5]. Now $D = |\mathcal{D}|$ for a subcomplex $\mathcal{D}$ of $|\Delta|$. Then the leg-graph (V, E) of $\mathcal{D}$ is a planar graph each of whose bounded face is a 4-cycle. Let V_C be the set of vertices belonging to (boundary cycle) C, and let F be the set of bounded faces of (V, E). By $|V_C| = |C|$, $4|F| = 2|E| - |C|$ and Euler formula $|V| - |E| + |F| = 1$, we obtain a combinatorial version of Gauss-Bonnet formula:

$$\sum_{x \in V \setminus V_C} (4 - \deg(x)) + \sum_{x \in V_C} (3 - \deg(x)) = 4,$$

where $\deg(x)$ denotes the degree of x (the number of edges incident to x) in (V, E). By the CAT(0) condition (2.3), *each inner node $x \in V \setminus V_C$ has degree at least* 4. Thus the first term of the LHS is nonpositive, and the second term is at least 4.

Now suppose that C is a 4-cycle, i.e., $|V_C| = 4$. Necessarily $(V, E) = (V_C, C)$. This means that D is a square, or the union of two triangles with a common hypotenuse. Thus we have (i). (ii) follows from (i). Indeed, the hypotenuse of 3-cycle belongs to some triangle. Use (i) for a possibly appearing 4-cycle of legs. □

In particular, if there is no folder consisting of at most two triangles, then Δ is completely reconstructed from its leg-graph; there are two ways splitting a square to two triangles. Moreover, for an arbitrary hereditary modular graph without $K_{3,3}$ and $K_{3,3}^-$ as induced subgraphs, by filling a folder to each maximal complete bipartite graph $K_{2,n}$ ($n \geq 2$), we obtain an F-complex without triangle-folders [3].

3.2 Geodesics

Here we investigate shortest paths connecting normal sets. Several important properties previously known for shortest paths connecting *vertices* can be naturally extended to shortest paths connecting *normal sets*.

Let Δ be an F-complex. In this subsection, *we assume that the leg-length κ is equal to* 1 *for simplicity.* Also keep in mind the fact that *the leg-graph Γ is bipartite* (since Γ is hereditary modular by the latter part of Theorem 3.2, and any hereditary modular graph is bipartite by Theorem 3.1). We begin with:

(3.7) For normal sets N and M, we have $d_\Delta(N, M) = d_\Gamma(N \cap V\Gamma, M \cap V\Gamma)$.

Indeed, we can easily modify a (polygonal) geodesic between N and M so that it lies on the union of legs (as in the proof of [11, Proposition 4.2]).

For notational simplicity, in the sequel we use the same d for d_Δ and d_Γ, and we denote $I_\Gamma(x, y)$ simply by $I(x, y)$. We will often use the following *bipartite property*:

Lemma 3.3. *For a normal set R and a leg $e = xy$, if $e \not\subseteq R$, then we have*

$$d(R, x) - d(R, y) \in \{-1, 1\}.$$

In particular, for a leg $e = xy$, if $x, y \in R$, then $e \in R$.

Proof. Let T be the closure of a connected component of $|\Delta| \setminus R$. Since R is connected and Δ is simply-connected, $T \cap R$ is connected. By the normality, $T \cap R$ consists of hypotenuses (or a single point). Therefore $T \cap R \cap V\Gamma$ is contained by one color class of bipartite graph Γ. The statement immediately follows from this fact. □

Lemma 3.4. *Let R be a normal set, and let p be a vertex in Δ. For a vertex $x \in R$, if $d(x, p) > d(R, p)$, then there exists an edge (leg or hypotenuse) $xy \subseteq R$ such that $d(x, p) = d(x, y) + d(y, p)$.*

Proof. We can take a vertex w in R with $d(w, p) = d(R, p)$ by (3.7). Since R is connected, we can take a path P in R connecting x and w so that P consists of legs and hypotenuses. Among such paths, take P with the following properties:

(a) the following sum I of $d(\cdot, p)$ along P is minimum:

$$I = \sum_{uv:\ \text{leg in } P} \frac{d(u, p) + d(v, p)}{2} + \sum_{uv:\ \text{hypotenuse in } P} d(u, p) + d(v, p).$$

(b) P contains as many legs as possible (under property (a)).

Let $P = (x = x_0, x_1, \ldots, x_m = w)$. Define a sequence $\{a_i\}_{i=0,\ldots,m}$ by $a_i = d(x_i, p)$. We show that $\{a_i\}_{i=0,\ldots,m}$ has no triple a_{i-1}, a_i, a_{i+1} with the property (i) $a_{i-1} = a_i > a_{i+1}$ or (ii) $a_{i-1} < a_i > a_{i+1}$. This implies the statement. Indeed, necessarily we have $a_0 > a_1$, implying $d(x, p) - d(x_1, p) \in \{1, 2\}$. By bipartiteness, edge xx_1 is a leg if $d(x, p) - d(x_1, p) = 1$, and xx_1 is a hypotenuse if $d(x, p) - d(x_1, p) = 2$.

Case (i). Suppose to the contrary that $a_{i-1} = a_i > a_{i+1}$ holds for some i. By the bipartite property, edge $x_{i-1}x_i$ is necessarily a hypotenuse. Then there is a triangle σ with vertices x_{i-1}, x_i, u such that $u \notin R$ and $d(u, p) = a_i - 1$. Indeed, we can take a triangle σ' with vertices x_{i-1}, x_i, u' (since every hypotenuse belongs to a triangle). By bipartiteness, $d(u', p) = a_i - 1$ or $a_i + 1$. Suppose $d(u', p) = a_i - 1$. Then $u' \notin R$. Otherwise $u'x_{i-1}, u'x_{i+1} \subseteq R$ by Lemma 3.3. Then $P \setminus \{x_{i-1}x_i, x_ix_{i+1}\} \cup \{x_{i-1}u', u'x_{i+1}\}$

has a smaller I than P; a contradiction to (a). Thus σ' is a triangle as required. So suppose $d(u',p) = a_i + 1$. By the quadrangle condition for u', x_{i-1}, x_i, p, there is a common neighbor w of x_{i-1}, x_i with $d(u',p) = 2 + d(w,p)$. As above, $w \notin R$. By (3.6) and normality of R, there is a triangle σ with vertices x_{i-1}, x_i, w, as required. Suppose that $x_i x_{i+1}$ is a leg. By the quadrangle condition for x_i, x_{i+1}, u, p, there are a common neighbor v of x_{i+1}, u and a folder F containing x_i, x_{i+1}, u, v (necessarily $u \neq p$). By normality, we have $v \in R$. Then there is a triangle σ' with vertices u, v, x_i in F such that M violates the local convexity at $\sigma \cup \sigma'$. A contradiction. Suppose that $x_i x_{i+1}$ is a hypotenuse. There is a triangle σ with vertices x_i, x_{i+1}, v such that $v \notin R$ (by construction (b)). If $v = u$, then R violates the local convexity on $\sigma \cup \sigma'$. So $v \neq u$. By the quadrangle condition for x_i, v, u, p, there is a common neighbor $z (\neq x_i)$ of v, u and a folder F supported by x_i, v, u, z. Again, by the quadrangle condition for v, z, x_{i+1}, N, there is a folder F' violating the flag condition at v with σ' and F. A contradiction.

Case (ii). Suppose to the contrary that $a_{i-1} < a_i > a_{i+1}$ holds for some i. Then at least one of $x_{i-1}x_i$ and $x_i x_{i+1}$ is a hypotenuse. Otherwise, by the quadrangle condition and the normality, the segment $x_{i-1}x_{i+1}$ belongs to M. This contradicts the construction (a). Suppose that both $x_{i-1}x_i$ and $x_i x_{i+1}$ are hypotenuses. Then there are triangles σ, σ' with vertices x_{i-1}, x_i, u and x_{i-1}, x_i, v (resp.) such that $u, v \notin M$. If $u = v$, then R violates the local convexity on $\sigma \cup \sigma'$. Therefore $u \neq v$. Also in this case, we can find a triple of folders violating the flag condition at u (or v) by repeated applications of the quadrangle condition, as above. Suppose that $x_{i-1}x_i$ is a hypotenuse and $x_i x_{i+1}$ is a leg. Then there is a triangle σ with vertices x_{i-1}, x_i, u such that $u \notin R$. Again repeated applications of the quadrangle condition imply the existence of a triple of folders violating the flag condition at u. A contradiction. □

In particular, any normal set R is geodesic, i.e., every points $p, q \in R$ can be joined by a path in R of length $d(p,q)$. The next lemma is an extension of the quadrangle condition (3.5). We also call it the quadrangle condition.

Lemma 3.5. *For a triple x, u, v of vertices and a normal set R, suppose that $d(x,u) = d(x,v) = 1$ and $d(u,R) = d(v,R) = d(x,R) - 1$.*

1. *If $d(x,R) = 1$, then there exists a triangle σ with vertices x, u, v such that $R \cap \sigma = uv$ (uv is a hypotenuse).*
2. *If $d(x,R) \geq 2$, then there exists a common neighbor w of u and v such that $d(w,R) = d(x,R) - 2$.*

Proof. (1). By Lemma 3.4, u and v are joined by a path P of length 2 in R. Apply (3.6) for cycle $P \cup \{ux, xv\}$. (2). So suppose that $d(x,R) = k \geq 2$. We use the induction on k. Take $u', v' \in R$ satisfying $d(u,u') = d(v,v') = k - 1$ with $d(u',v') (\leq 2k)$ minimum. Suppose $d(u',v') = 2k$. Then $x \in I(u',v')$. Let u'_0 be a neighbor of u' in $I(u',x)$. By Lemma 3.4, we can take a vertex u'_1 joined to u' by a leg or a hypotenuse in R so that $d(u',v') = d(u',u'_1) + d(u'_1,v')$. Suppose that $u'u'_1$ is a leg. By the quadrangle condition for u', u'_0, u'_1, v', there is a common neighbor r of u'_0, u'_1 with $d(u',v') = d(r,v') + 2$. By the normality and (3.6), there is a triangle σ with vertices u', u'_1, r and hypotenuse $u'r = R \cap \sigma$. Then $r \in R$, $d(r,v') = d(u',v') - 2$, and $d(r,u) = d(u',u) = k - 1$; this contradicts the minimality assumption of (u',v'). So suppose that $u'u'_1$ is a hypotenuse. Take a triangle σ with hypotenuse $u'u'_1$ and right angle r. If $r = u'_0$, then this contradicts the minimality of u', as above. So $r \neq u'_0$. By the quadrangle condition for u', u'_0, r, v', there are a common neighbor r' of u'_0, r with $d(u',v') = d(r',v') + 2$, and a folder F containing u', u'_0, r, r' by (3.6). By minimality assumption, $r' \neq u'_1$. Again by the quadrangle condition for r, r', u'_1, v' we obtain a folder F' violating the flag condition (2.3) with F and σ at r; a contradiction.

So suppose $d(u', v') < 2k$. Take a median $w^*(\neq x)$ of u', v', x and take a neighbor w of x from $I(x, w^*)$. If $w = v$, then by the quadrangle condition for x, u, v, u' we can find a vertex as required. Therefore $w \neq u, v$. By the quadrangle condition for x, u, w, u', we can find a common neighbor u^* of u, w with $d(u^*, u') = d(u^*, R) = k - 2$. Similarly we can find a common neighbor v^* of w, v with $d(v^*, v') = d(v^*, R) = k - 2$. If $u^* = v^*$, then we are done. So $u^* \neq v^*$. Let F_1 and F_2 be the folders containing x, u, w, u^* and x, v, w, v^*, respectively. Apply the induction to quadrangle w, u^*, v^*, R. Then we can find a folder F_3 violating the flag condition at w with F_1 and F_2. A contradiction. □

In the proof of Lemma 3.4 we can replace (R, p) by a pair of normal set (M, N), and can use our new quadrangle condition (Lemma 3.5) instead of (3.5). Thus we obtain:

Lemma 3.6. *Let M, N be two normal sets. For a vertex $x \in M$, if $d(x, N) > d(M, N)$, then there exists an edge (leg or hypotenuse) $xy \subseteq M$ such that $d(x, N) = d(x, y) + d(y, N)$.*

From the proof we also obtain the following (we can choose P as a shortest path):

Lemma 3.7. *Let M, N be two normal sets. For two distinct vertices $x, y \in M$ with $d(x, N) = d(y, N) = d(M, N)$, there is a hypotenuse xz in M with $d(z, y) = d(x, y) - 2$ and $d(z, N) = d(M, N)$.*

Next we study the extendability of a path to a shortest path connecting normal sets. An F-complex Δ is called *star-shaped* if there is a vertex p such that every folder contains p and no triangle contains p as its right angle. In this case, the CAT(0) condition (2.3) can be rephrased as the girth condition of the boundary leg-graph:

(3.8) $\Gamma \setminus p$ is a bipartite graph with girth at least 8.

For a vertex p in Δ, let Δ_p be the subcomplex of Δ consisting of cells containing p and their faces. Clearly Δ_p is also an F-complex. Although Δ_p may not be star-shaped, $(\Delta^2)_p$ is always star-shaped.

Lemma 3.8. *Let p be a vertex, and let R be a normal set not containing p. Suppose that Δ_p is star-shaped. Then a point x in $|\Delta_p|$ at minimum distance from R is a vertex and is uniquely determined.*

We call this vertex x the *gate* of R in Δ_p; this definition is compatible with that in [7].

Proof. By $p \notin R$ and (2.4) (i), R does not contain a leg in Δ_p. Since Δ_p is star-shaped, the boundary (relative to $|\Delta|$) consists of legs. By Lemma 3.3, the map $x \mapsto d(x, R)$ on a leg is monotone decreasing or increasing. Hence the minimum attains at a vertex. Suppose that two vertices x, y have the minimum distance from R. Consider the case where x and y are not incident to p and have a common neighbor v ($\in \Delta_p$). Then x, v, p and y, v, p belong to folders F_1 and F_2, respectively. By the quadrangle condition for v, x, y, R, we obtain a folder F_3 violating the flag condition with F_1 and F_2. A contradiction. Consider the other cases. Take a path P connecting x and y in Δ_p passing p. By applying the quadrangle condition for three consecutive vertices in P and R, we find a vertex u in Δ_p with $d(u, R) < d(x, R)$, or a vertex z in Δ_p with $d(z, R) = d(x, R)$ such that z and x have a common neighbor. The first case is clearly impossible, and the second case reduces to the case above for (x, z). □

The extendability of a shortest path passing a vertex p can be locally characterized by the gates in Δ_p (or $(\Delta^2)_p$ when Δ_p is not star-shaped).

Lemma 3.9. *Let p be a vertex, and let M, N be normal sets not containing p. Suppose that Δ_p is star-shaped. Let x and y be the gates of M and N in Δ_p, respectively. Then the following conditions are equivalent:*

(1) $d(M,N) = d(M,x) + d(x,p) + d(p,y) + d(y,N)$.

(2) $d(x,y) = d(x,p) + d(p,y)$.

(3) *There is no leg pq with $x, y \in B(q,1)$.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious. We show (3) $\Rightarrow$ (2). By $d(x,p), d(p,y) \in \{1,2\}$ and bipartiteness of Γ, $d(x,y) < d(x,p) + d(p,y)$ implies $(d(x,y), d(x,p), d(p,y)) \in \{(0,1,1),(1,1,2),(1,2,1),(0,2,2),(2,2,2)\}$. For the first three cases, we can take q as x or y. For the fourth case, we can take q as any common neighbor of p, x. For the last case, take q as a median of x, y, p; then $d(q,x) = d(q,y) = d(q,p) = 1$ as required.

Next we show (2) $\Rightarrow$ (1) or (3) $\Rightarrow$ (1). We first consider the case where both M and N are singletons, say $M = \{u\}$ and $N = \{v\}$. Suppose $d(u,v) < d(u,x) + d(x,p) + d(p,y) + d(y,v)$. This implies $d(u,v) < d(u,p) + d(p,v)$ by $d(u,p) = d(u,x) + d(x,p)$ and $d(v,p) = d(v,y) + d(y,p)$. Take a median w of u, v, p. Necessarily $w \neq p$. So we can take a vertex $q \in I(p,w)$ incident to p by a leg. We show $x, y \in B(q,1)$. Suppose $x \notin B(q,1)$. Take vertex $q' \in I(p,x)$ incident to p by a leg (possibly $q' = x$). Then $d(q',u) = d(q,u) = d(p,u) - 1$. Apply the quadrangle condition for p, q, q', u. We obtain a vertex q^* in Δ_p such that $d(q^*,u) \leq d(x,u)$ and $q^* \neq x$; a contradiction of definition and uniqueness of gate x. Thus $x, y \in B(q,1)$. Next consider the case where M is general and N is a singleton $\{v\}$. Suppose (indirectly) that (2) holds and (1) fails. Take a vertex $u \in M$ with $d(u,x) = d(M,x) = d(M,|\Delta_p|)$ so that $d(u,v)$ is minimum. Now x is also the gate of u in Δ_p. By the singleton case above, we have $d(u,v) = d(u,x) + d(x,p) + d(p,y) + d(y,v)$. Thus $d(u,v) > d(M,v)$. By Lemma 3.4, we can take an edge uw in M such that $d(u,v) = d(u,w) + d(w,v)$. Take a vertex $u' \in I(u,x) \cup I(x,p)$ incident to u. Of course $u' \notin M$, and $d(u,v) = d(u',v)+1$. Suppose that uw is a leg. By the quadrangle condition for u, w, u', v and by the normality of M, we can find a triangle σ with vertices u, u', z and hypotenuse $uz = \sigma \cap M$ such that $d(z,v) = d(u,v) - 2$. Then $d(z,x) = d(u,x)$. However this is a contradiction to minimality of u. Suppose that uw is a hypotenuse; $d(u,v) = d(w,v) + 2$. We can take a triangle σ with hypotenuse uw and right angle z. If $z \in M$, then $uz \in M$, and apply the above argument. So $z \notin M$. If $z = u'$, then $d(w,x) = d(u,x)$, and this contradicts to the minimality of u as above. So $u' \neq z$. By the repeated applications of the quadrangle condition (for u, u', z, v and for $z, \cdot, w, v$), we find a violating triple of folders at z; a contradiction. Finally we consider the case where both M and N are general. The argument is essentially same as above; suppose that (2) holds and (1) fails. Take a vertex $u \in M$ with $d(u,x) = d(M,x) = d(M,|\Delta_p|)$ so that $d(u,N)$ is minimum. Now x is also the gate of u in Δ_p. By the case above, we have $d(u,N) = d(u,x) + d(x,p) + d(p,y) + d(y,N)$. Thus $d(u,N) > d(M,N)$. By Lemma 3.6, we can take an edge uw in M such that $d(u,v) = d(u,w) + d(w,v)$. We can apply the same argument above, which leads a contradiction. □

By the same idea as in the proof of Lemma 3.8, we have:

Lemma 3.10. *For a folder F and a normal set R not meeting the interior of F, a point x in F at minimum distance from R is a vertex and is uniquely determined.*

Also we call x the *gate* of R in F. Since $F = |(\mathit{\Delta}^2)_p|$ for the center vertex p of the 2-subdivision of F, as a corollary of Lemma 3.9, we obtain an extension of [17, p.87, Claim 3], where N and M were assumed to be singletons.

Lemma 3.11. *Let F be a folder, and M, N normal sets not meeting the interior of F. Let x and y be the gates of M and N in F, respectively. If x and y are not adjacent, then we have*

$$d(M, N) = d(M, x) + 2 + d(y, N).$$

We also give a variant of Lemma 3.11; a proof is similar.

Lemma 3.12. *Let M, N be normal sets, and let σ be a triangle with vertices x, y, z and hypotenuse yz. Suppose that $M \cap \sigma = yz$ and x is the gate of N in the folder containing σ. Then we have*

$$d(M, N) = 1 + d(x, N).$$

3.3 Helly property of normal sets

The goal of this subsection is to prove:

Theorem 3.13. *For an F-complex, the collection of normal sets has the Helly property.*

In fact, (3.1) (i) is a simple corollary of it, based on the following (3.9). Consider a normal set N, a rational radius r, and the corresponding ball $B(N, r)$. We can subdivide $\mathit{\Delta}$ into $\mathit{\Delta}^k$ and further subdivide $\mathit{\Delta}^k$ into $\tilde{\mathit{\Delta}^k}$ by splitting squares into two triangles so that $B(N, r)$ is the union of a subcomplex of $\tilde{\mathit{\Delta}^k}$.

(3.9) $B(N, r)$ is normal in $\tilde{\mathit{\Delta}^k}$.

Proof. Note that N is also normal in $\tilde{\mathit{\Delta}^k}$. First we show (2.4) (i). Take a leg $e = uv$ in $B(N, r)$ and not in N. By the bipartite property, we may assume $d(u, N) = d(v, N) + 1/k$. Take a cell σ containing e. If σ has a vertex w with $d(w, N) \leq r - 2/k$, then obviously $\sigma \subseteq B(N, r)$ since every point in σ has distance at most $2/k$ from w. We may assume that such a vertex does not exist. Thus $d(u, N) = r$. If v is incident to vertex $w (\neq u)$ in σ by a leg, then $d(w, N) = r$, σ is a triangle with vertices u, v, w (by construction), and hence $\sigma \subseteq B(N, r)$. Suppose that v is incident to vertex $w (\neq u)$ in σ by a hypotenuse. Necessarily $d(w, N) = d(u, N) = r - 1/k$ (and $\sigma \subseteq B(N, r)$). Indeed, if $d(w, N) = r + 1/k$, then $\sigma \cap B(N, r) \neq \emptyset$ and $\sigma \setminus B(N, r) \neq \emptyset$, contradicting a construction; we need more subdivisions.

Next we verify the local convexity. Suppose to the contrary that there are adjacent two triangles σ, σ' violating (2.4) (ii) with $B(N, r)$. Let x be a common right angle of σ and σ'. Then $d(x, N) = r + 1/k$. Let y and z be distinct neighbors of x belonging to σ and σ', respectively. Then $d(y, N) = d(z, N) = r$. By the quadrangle condition for x, y, z, N, we can find a folder F violating the flag condition at x with σ and σ'. A contradiction. $\square$

We prove Theorem 3.13. Let $\{N_i\}_{i \in I}$ be a pairwise intersecting family of normal sets in $\mathit{\Delta}$; in particular $d(N_i, N_j) = 0$ for $i, j \in I$. Consider $\mathit{\Delta}^2$. Take a vertex p in $\mathit{\Delta}^2$ such that $D(p) := \max_{i \in I} d(p, N_i)$ is minimum. We show $D(p) = 0$. Namely p is a common point as required. Suppose indirectly $D(p) > 0$. Since $q \mapsto d(q, N_i)$ is a linear function on each leg in $\mathit{\Delta}$ (by the bipartite property), we may assume that p is a vertex in $\mathit{\Delta}$ or the midpoint of some folder in $\mathit{\Delta}$. In particular $(\mathit{\Delta}^2)_p$ is star-shaped. Let I_0 be the (nonempty) set of indices $i \in I$ with $p \notin N_i$. For each $i \in I_0$, the gate g_i of N_i in $(\varDelta^2)_p$ is well-defined. We claim:

(3.10) For any $i \in I_0, j \in I \setminus I_0$, the gate g_i is contained by N_j.

Proof. Now $d(g_i, p) \in \{1, 1/2\}$; note that the leg-length of $\varDelta^2$ is 1/2. Suppose first $d(g_i, p) = 1$, i.e., $\{g_i, p\}$ is a nonadjacent pair of a folder F in $\varDelta^2$. Obviously g_i is the gate of N_i in F. If $F \cap N_j = \{p\}$, then p is the gate of N_j in F, and Lemma 3.11 implies a contradiction $0 = d(N_i, N_j) = d(N_i, g_i) + 1 + d(p, N_j) > 0$. So $F \cap N_i \supset \{p\}$ (proper inclusion). By normality, N_i includes g_i.

Next suppose $d(g_i, p) = 1/2$, i.e., g_i is incident to p by a leg. Suppose indirectly $g_i \notin N_j$. Consider $\varDelta^8$. Take a vertex $q \in pg_i$ in $\varDelta^8$ incident to p by a leg. Consider $(\varDelta^8)_q$ (star-shaped). Then g_i is also the gate of N_i in $(\varDelta^8)_q$. Consider the gate g of N_j in $(\varDelta^8)_q$. Then $g \neq g_i$, and g is not incident to g_i; thus (3) in Lemma 3.9 holds. Indeed, since every cell in $\varDelta^2$ containing g_i does not meet N_j in its interior, for any point g' with $d(g_i, g') \leq 1/8$ we have $d(g', N_j) \geq 3/8$. On the other hand, $d(|(\varDelta^8)_q|, N_j) \leq d(|(\varDelta^8)_q|, p) = 1/4$. By (3) $\Rightarrow$ (1) in Lemma 3.9 we get a contradiction $0 = d(N_i, N_j) = d(N_i, g_i) + d(g_i, q) + d(q, g) + d(g, N_j) > 0$. □

Suppose that there is $i \in I_0$ such that g_i is incident to p in $\varDelta^2$. For each $j \in I_0$, $g_j = g_i$ or g_j is incident to g_i by a leg in $(\varDelta^2)_p$. Otherwise Lemma 3.9 (3) $\Rightarrow$ (1) implies a contradiction $0 = d(N_i, N_j) = d(N_i, g_i) + d(g_i, p) + d(p, g_j) + d(g_j, N_j) > 0$. In particular $d(g_j, N_j) = d(p, N_j) - 1/2$ for $j \in I_0$. So $D(g_i) < D(p)$; a contradiction to the minimality.

So suppose that for each $i \in I_0$, the gate g_i is not incident to p; $d(g_i, p) = 1$. If $g_i = g_{i'}$ for all $i, i' \in I_0$, then $D(g_i) < D(p)$; a contradiction to the minimality. So suppose that $g_i \neq g_{i'}$ for some $i, i' \in I_0$. Necessarily g_i and $g_{i'}$ have a common neighbor q; otherwise $0 = d(N_i, N_{i'}) = d(N_i, g_i) + d(g_i, p) + d(p, g_{i'}) + d(g_{i'}, N_{i'}) > 0$ by Lemma 3.9. By the girth condition (3.8), q is incident to g_i for all $i \in I_0$. By normality, for all $j \in I \setminus I_0$, N_j includes q. Then $D(q) < D(p)$; a contradiction to the minimality assumption.

3.4 Orbits

We recall the notion of orbits [17, 18] with a slight modification by [11]. Edges $e, e' \in E\varGamma$ are called *mates* if there exists a square σ in $\varDelta$ containing e, e' as nonincident edges or there exists a folder F consisting of triangles such that F contains both e, e' as legs. Edges e, e' are *projective* if there is a sequence $e = e_0, e_1, \ldots, e_m = e'$ such that e_i and e_{i+1} are mates. The projectiveness defines an equivalence relation on $E\varGamma$. An equivalence class is called an *orbit*. The set of all orbits is denoted by $\mathcal{O} = \mathcal{O}^{\varDelta}$. The following property is an extension of [18, Statement 2.2].

Proposition 3.14. *Let M, N be normal sets. Let P, Q be paths in $\varGamma$ connecting M and N. If P is a shortest path between M and N, then $|P \cap O| \leq |Q \cap O|$ holds for each orbit $O \in \mathcal{O}$.*

Proof. We first consider the case where M is a singleton set $\{x\}$. We use the induction on the length k of Q. We may assume that $k \geq 1$ and the length of P is also positive, and assume that vertices u and v following x in P and Q (resp.) are distinct. By the bipartite property (Lemma 3.3), $d(x, N) - d(v, N) \in \{1, -1\}$. Suppose $d(x, N) - d(v, N) = -1$. Then $P \cup \{xv\}$ is a shortest path between v and N, and $Q \setminus \{xv\}$ is a path between v and N of length $k - 1$. By induction we have $|Q \cap O| \geq |(Q \setminus \{xv\}) \cap O| \geq |(P \cup \{xv\}) \cap O| \geq |P \cap O|$. Suppose $d(x, N) - d(v, N) = 1$. Suppose $|P| = 1$. By the quadrangle condition (Lemma 3.5), there is a triangle σ with vertices x, u, v with $N \cap \sigma = uv$. Therefore

xu and xv belong to the same orbit, and $|P \cap O| \leq |Q \cap O|$ is obvious. Suppose $|Q| \geq |P| \geq 2$. By the quadrangle condition for x, u, v, N, we can find a common neighbor w of u, v with $d(w, N) = d(x, N) - 2$. We can take a shortest path P^* connecting w and N. Clearly $P^* \cup \{uw\}$ is a shortest path connecting u and N. By the induction, $|(P \setminus \{xu\}) \cap O| = |(P^* \cup \{uw\}) \cap O|$. Similarly, $|(Q \setminus \{xv\}) \cap O| \geq |(P^* \cup \{vw\}) \cap O|$. Since xu and vw are mates and xv and uw are mates, we have $|P \cap O| = |(P^* \cup \{wu, ux\}) \cap O| = |(P^* \cup \{wv, vx\}) \cap O| \leq |Q \cap O|$.

Suppose that M is not a singleton set. Let x and y be the ends of P and Q belonging to M, respectively. By Lemma 3.6 and the singleton case above, it suffices to show the equality $|Q \cap O| = |P \cap O|$ for the case where Q is also shortest. By above, we may assume $x \neq y$. We use the induction on $d(x, y)$. By Lemma 3.7, we can take $x' \in M$ joined to x by a hypotenuse with $d(x', N) = d(x, N)$ and $d(x', y) = d(x, y) - 2$. There is a triangle with hypotenuse xx' and right angle w such that $d(w, N) = d(x, N) - 1$. Take any shortest path $\tilde{P}$ connecting w and M. Then we have $|P \cap O| = |(\tilde{P} \cup \{xw\}) \cap O| = |(\tilde{P} \cup \{x'w\}) \cap O| = |Q \cap O|$, where the last equality follows from the induction. □

For a (disjoint) union U of orbits, we can construct a new 2-complex Δ^U from Δ by identifying the ends of all edges not belonging to U. Namely regard Δ as an abstract 2-complex $(\Gamma_0; \mathcal{C})$ with property (2.2). Contract all edges not in U in Γ_0 and delete parallel edges and loops (of hypotenuses) appeared, and also delete 1-cycles (loops) and 2-cycles from $\mathcal{C}$. It will turn out soon that the resulting $({\Gamma_0}^U; \mathcal{C}^U)$ fulfills (2.2). Thus we obtain the 2-complex Δ^U with (2.1). The leg-graph of Δ^U is denoted by Γ^U. This construction naturally induces the map $(\cdot)^U : V\Gamma \to V\Gamma^U$ by defining x^U to be the contracted vertex. In particular, if $xy = e \in U$, then $x^U \neq y^U$ and $x^U y^U \in E\Gamma^U$. Indeed, $\{e\}$ is a shortest path between x and y. Then $x^U = y^U$ implies that there is a path Q in W between x and y. This implies $0 = |Q \cap U| < |\{e\} \cap U| = 1$; a contradiction to Proposition 3.14. Similarly, if all legs of a folder F belong to U, then $(\cdot)^U$ is injective on the vertex set of F. Thus, by extending linearly, we obtain map $(\cdot)^U : |\Delta| \to |\Delta^U|$. In particular, if all legs of a folder F belong to U, then $(\cdot)^U$ is injective on F.

Proposition 3.15. *Let Δ be an F-complex, and let U be the union of some orbits, and let $W = E\Gamma \setminus U$.*

(1) *Δ^U is an F-complex, and if Δ is orientable, then so is Δ^U.*

(2) *For a normal set R in Δ, R^U is also normal in Δ^U.*

(3) *For a shortest path P in Γ connecting normal sets M and N, P^U is also a shortest path in Γ^U connecting M^U and N^U. In particular, we have*

$$d_\Delta(M, N) = d_{\Delta^U}(M^U, N^U) + d_{\Delta^W}(M^W, N^W).$$

Proof. (1). We need to verify (2.2) (before (2.3)). From the definition of orbits and the fact that $(\cdot)^U$ is injective on a cell each of whose legs belongs to U, we see that every hypotenuse in Δ^U belongs a triangle. We next show that for any two folders F_1, F_2 all of whose legs belong to U, the intersection ${F_1}^U \cap {F_2}^U$ is a vertex or a leg or empty; if true, then Δ^U (or $({\Gamma_0}^U; \mathcal{C}^U)$) fulfills (2.2) and (2.3) (i). Suppose that ${F_1}^U \cap {F_2}^U$ is nonempty. So there are vertices $x \in F_1, y \in F_2$ with $x^U = y^U$. We claim:

(∗) For any vertex x' in F_1 we have $d(x', y) = d(x', x) + d(x, y)$.

Otherwise we can find a shortest path P connecting x and y using a leg e in F_1. However by $x^U = y^U$ there is a path Q in W between x and y. Then $|Q \cap U| = 0 < 1 \leq |P \cap U|$; a contradiction to Proposition 3.14.

Suppose indirectly that there is a vertex $z(\neq x)$ in F_1 not incident to x with $z^U \in {F_2}^U$. Take a vertex w in F_2 with $w^U = z^U$. We may assume $x \neq y$. So we can take a leg $xu \notin F_1$ with $d(x,y) = 1 + d(u,y)$. By $(*)$ we have $d(x,w) = 1 + d(u,y) + d(y,w) = 1 + d(u,w)$. Also by $(*)$ we have $d(x,w) = d(x,z) + d(z,w)$. Take a common neighbor v of x,z in F_1. By the quadrangle condition for x,u,v,w we obtain a common neighbor t of u,v with $d(t,w) = d(x,w) - 2$ and a folder $F(\neq F_1)$ containing x,u,v,t. Again by the quadrangle condition for v,t,z,w we obtain a folder F' violating the flag condition with F_1 and F at v; a contradiction. Thus ${F_1}^U \cap {F_2}^U$ is a vertex or a leg.

Next we show that Δ^U fulfills the flag condition (2.3) (ii). Suppose not. Then there are three folders F_1, F_2, F_3 such that each folder F_i contains legs $r_i x_{i,j}$ in U ($j \in \{1,2,3\} \setminus i$), and ${r_i}^U = {r_j}^U$ and ${x_{i,j}}^U = {x_{j,i}}^U$ for $i,j \in \{1,2,3\}$. By subdivision, we may assume that each F_i is not a triangle-folder. Here $d(x_{2,3}, x_{3,2}) < d(x_{2,3}, r_1) + d(r_1, x_{3,2})$ holds. Otherwise by $(*)$ we can take a shortest path between $x_{2,3}$ and $x_{3,2}$ using $r_2 x_{2,3} \in U$. However $x_{2,3}$ and $x_{3,2}$ are connected by a path in W by ${x_{2,3}}^U = {x_{3,2}}^U$. This contradicts Proposition 3.14. Take a median w of $r_1, x_{2,3}, x_{3,2}$. Then $w \neq r_1$. We can take a vertex u incident to r_1 with $d(w, r_1) = 1 + d(u,w)$. Then $d(r_1, x_{2,3}) = 1 + d(u, x_{2,3}) = 1 + d(r_1, r_2)$. Here u does not belong to F_1. Otherwise u and $x_{2,3}$ are connected by a path in W, since any shortest path uses exactly one leg in U by $(*)$ and Proposition 3.14. Then ${x_{2,3}}^U = u^U \in {F_1}^U$; a contradiction to ${F_1}^U \cap {F_2}^U = {r_2}^U {x_{2,1}}^U$. By $(*)$, we have $d(r_1, x_{2,1}) = 1 + d(r_1, r_2) = 1 + d(x_{1,2}, x_{2,1})$. Hence $d(r_1, r_2) = d(x_{1,2}, x_{2,1}) = d(u, x_{2,3})$. Take a vertex x_2 in F_2 not incident to r_2. By $d(r_1, x_2) = d(r_1, r_2) + 2$ and $d(x_{1,2}, x_2) = d(x_{1,2}, x_{2,1}) + 1$, we have $d(r_1, x_2) = 1 + d(x_{1,2}, x_2) = 1 + d(u, x_2)$. By the quadrangle condition for $r_1, x_{1,2}, u, x_2$, there is a folder $F' \neq F_1$ containing $r_1, u, x_{1,2}$. Similarly there is a folder $F' \neq F_1, F'$ containing $r_1, u, x_{1,3}$. Thus F_1, F', F'' violate the flag condition at r_1; a contradiction.

We next verify that Δ^U is simply-connected. Take a cycle C in the 1-skeleton graph of Δ^U. We show that C is contractible to one point in Δ^U. We may assume that C consists of legs (since every hypotenuse belongs to a triangle). Then we can take a cycle C_0 in Γ with ${C_0}^U = C$. Fix a vertex x in C_0. Take a vertex $y \neq x$ in C_0 with maximum distance from x. Let y', y'' be the neighbors of y in C_0. Then $d(y,x) = 1 + d(y',x) = 1 + d(y'',x)$. If $y' = y''$, then delete leg yy' from C_0. If $y' \neq y''$, then by the quadrangle condition there is a common neighbor z of y', y'' with $d(x,z) = d(x,y) - 2$, and modify C_0 by deleting yy', yy'' and adding $y'z, y''z$. Repeat it until C_0 become one point x. Now consider this process in Δ^U by $C_0 \mapsto (C_0)^U$. In the change above, y, y', y'', z belong to a folder in Δ, and hence y, y', y'', z^U belong to a folder or a leg of Δ^U. This means that the change can be done continuously in Δ^U. So C is contractible to x^U in Δ^U.

The orientability statement can be seen as follows. Suppose that two legs xy, uv belong to U with $(x^U, y^U) = (u^U, v^U)$. By the same argument for $(*)$, we have $d(x,v) = d(x,y) + d(y,v)$. So $d(x,u) = d(y,v)$. By repeat applications of the quadrangle condition, we get shortest paths $(x, x_1, x_2, \ldots, u)$ and $(y, y_1, y_2, \ldots, v)$ such that x_i is incident to y_i. Therefore, if xy is oriented as $\overrightarrow{xy}$, then uv is oriented as $\overrightarrow{uv}$. Thus an orientation of Δ induces an orientation of Δ^U.

(3). Suppose that P^U is not shortest. Then there is a shorter path P' in Γ^U. By adding edges not in U to P', we obtain a (not necessarily shortest) path P^* in Γ connecting M and N. Then there is an orbit $O \subseteq U$ such that $|O \cap P^*| < |O \cap P|$. By Proposition 3.14, P is not shortest. A contradiction.

(2). It suffices to verify the local convexity condition. Suppose (indirectly) that there are a normal set R and two triangles σ_1, σ_2 such that R^U violates the local convexity on the union of (bijective images) ${\sigma_1}^U$ and ${\sigma_2}^U$. Suppose that σ_i has vertices x_i, y_i, r_i with hypotenuse $x_i y_i = R \cap \sigma_i$ ($i = 1,2$), and that $({r_1}^U, {x_1}^U) = ({r_2}^U, {x_2}^U)$. Consider

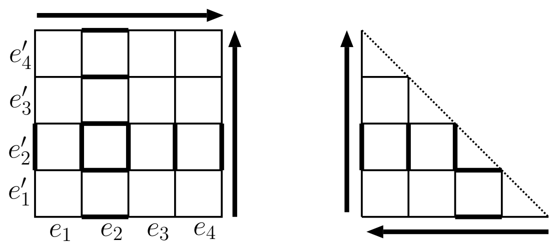

Figure 9: the construction of U_i; the bold lines represent U_2

the gate g of r_1 in the folder containing σ_2. If $g = r_2$, then by Lemma 3.12 we have $d(r_1, R) = d(r_1, g) + 1 > 1 = d(r_1, R)$. Therefore $g \neq r_2$. Take a shortest path P from r_1 to r_2 including g, and consider its image P^U, which is a shortest path connecting ${r_1}^U$ and ${r_2}^U$ in Γ^U by (3). However its length is at least 1. A contradiction to ${r_1}^U = {r_2}^U$. □

Now we are ready to prove (3.1) (ii). For a map $\rho : X \to |\Delta|$, we define metric d^ρ on X by

$$d^\rho(x, y) = d_\Delta(\rho(x), \rho(y)) \quad (x, y \in X). \tag{3.11}$$

Proposition 3.16. *Let Δ be an orientable F-complex, and let Γ be its leg-graph. For any map $\rho : X \to |\Delta|$ on a finite set X, there exist maps $\rho_i : X \to V\Gamma$ $(i \in I)$ such that d^ρ is a convex combination of d^{ρ_i}, and for any normal set R, $\rho(x) \in R$ implies $\rho_i(x) \in R$ $(i \in I)$.*

We may consider the case where $\rho(X) \subseteq V\Gamma^k$ for sufficiently large k. Each edge e of Γ is subdivided into k edges $e_1, e_2, \ldots, e_k$. Fix an orientation of Γ with property (2.5). Suppose that $(e_1, e_2, \ldots, e_k)$ is arranged from the source to the sink in the orientation of e. Let U_i be the set of edges projective to i-th edge e_i. By construction, $U_i \cap U_j = \emptyset$ for $i \neq j$, and U_i is the union of orbits of Δ^k; see Figure 9. By Proposition 3.15 (3), we have

$$d_\Gamma(p, q) = \sum_{i=1}^{k} d_{(\Gamma^k)^{U_i}}(p^{U_i}, q^{U_i}) \quad (p, q \in V\Gamma).$$

Each Δ^{U_i} is isomorphic to Δ up to dilation factor of $1/k$. Therefore $p \mapsto p^{U_i}$ induces the map $\phi_i : |\Delta| \to V\Gamma$. Then $\phi_i \circ \rho$ $(i = 1, 2, \ldots, k)$ are the desired maps. Also, by construction, $\rho(x) \in \sigma$ implies $\rho_i(x) \in \sigma$ for any cell $\sigma \in \Delta$. This implies the latter statement.

4 Related results

In this section, we present two related results. The first one is an optimality criterion of a combinatorial dual problem (2.7), which roughly characterizes the adjacency of extreme points of the feasible region in LP-dual (3.2). This result will play a crucial role in the splitting-off argument [13]. The second one is a characterization of distances of combinatorial dimension at most 2 in terms of F-complex realizations.

4.1 An optimality criterion for combinatorial dual problems

Here we show that (2.7) has an optimality criterion of the following type:

> *If ρ is not optimal, then there exists another ρ' close to ρ having the smaller objective value.*

We will characterize the closeness in terms of orientations of Γ. An orientation of Γ with property (2.5) is said to be *admissible*. Admissible orientations are obtained by orienting orbits independently. Such an orientation of an orbit is also said to be *admissible*. Each orbit has exactly two admissible orientations; one is the reverse of another. A map ρ feasible to (2.7) is called a *potential*.

Consider an orbit $\overrightarrow{O}$ oriented according to some admissible orientation. A potential ρ' is called a *neighbor* of a potential ρ with respect to $\overrightarrow{O}$ if for each $x \in VG$ with $\rho'(x) \neq \rho(x)$

(4.1) (i) there is an oriented edge $\overrightarrow{pq} \in \overrightarrow{O}$ such that $\rho(x) = p$ and $\rho'(x) = q$, or

(ii) there are two oriented edges $\overrightarrow{pq}, \overrightarrow{qr} \in \overrightarrow{O}$ belonging to a common 2-cell in Δ such that $\rho(x) = p$ and $\rho'(x) = r$.

The main theorem here is:

Theorem 4.1. *If a potential ρ is not optimal, then there exists a neighbor ρ' of ρ having the smaller objective value.*

Remark 4.2. The combinatorial dual problem (2.7) is a variant of *multifacility location problem* on graph Γ [24]. From this point of the view, Theorem 4.1 can be understood as a generalization of a result of Kolen in tree location problems (the case where Γ is a tree); see [21, Chapter 3].

The proof is based on the idea used in [12]. A map ρ feasible to (2.6) is called a *continuous potential*. We denote the distance between points $\rho(x)$ by d^ρ as (3.11), and also denote the corresponding objective value $\sum_{xy \in EG} c(xy) d_\Gamma(\rho(x), \rho(y))$ simply by $\langle c, d^\rho \rangle$.

Recall the correspondence between metrics m feasible to (3.2) and a continuous potentials ρ, discussed in the beginning of Section 3. Namely, for a minimal feasible metric m, there is a continuous potential ρ satisfying $m = d^\rho$. Here a feasible metric m is said to *minimal* if there is no other feasible metric $m' (\neq m)$ with $m' \leq m$. In general, this correspondence $m \mapsto \rho$ is not one-to-one. However one can establish a one-to-one correspondence by the following trick. Add isolated terminals to S and extend μ so that $\{R_s\}_{s \in S}$ contains all singleton sets of $V\Gamma$. In this case, the correspondence $m \mapsto \rho$ is one-to-one, and is continuous. Suppose that for a minimal feasible metric m there are two continuous potentials ρ, ρ' with $m = d^\rho = d^{\rho'}$. Suppose $\rho(x) \neq \rho'(x)$ for some $x \in VG$. Take minimal cells σ and σ' of Δ containing $\rho(x)$ and $\rho'(x)$, respectively. Suppose that σ and σ' are distinct. Then there is a vertex v in σ such that $d_\Delta(v, \rho(x)) \neq d_\Delta(v, \rho'(x))$. By the hypothesis, there is a terminal s with $R_s = \{v\}$. Then $\rho(s) = \rho'(s) = v$ and $d^\rho(s, x) = d_\Gamma(v, \rho(x)) \neq d_\Gamma(v, \rho'(x)) = d^{\rho'}(s, x)$. A contradiction. Therefore $\sigma = \sigma'$. Since distances $d_\Delta(v, \rho(x)) (= d_\Delta(v, \rho'(x)))$ from vertices v of σ uniquely determine $\rho(x)$, two maps ρ and ρ' must coincide.

We are ready to prove the statement. We remark that if the image of ρ belongs to $V\Gamma$, then d^ρ is minimal; this also follows from the fact that $\{R_s\}_{s \in S}$ contains all singleton sets. Suppose that a potential ρ is not optimal. Equivalently, d^ρ is not optimal to (3.2). Since d^ρ is minimal and nonoptimal, there is another minimal feasible metric m' sufficiently close to d^ρ such that $\langle c, m' \rangle < \langle c, d^\rho \rangle$. By the minimality, m' is represented by $d^{\rho'}$ for a continuous potential ρ'. Since m' is sufficiently close to d^ρ and $m' \mapsto \rho'$ is continuous, we may assume that

$$d_\Delta(\rho(x), \rho'(x)) < \kappa/2 \quad (x \in VG). \tag{4.2}$$

Recall that κ is the leg-length of Δ. Furthermore we may assume that $\rho(VG) \subseteq V\Gamma^k$ for sufficiently large k. Fix some admissible orientation $\overrightarrow{\Gamma}$ of Γ. Decompose $E\Gamma^k$ into $U_1, U_2, \ldots, U_k$ and define maps $\phi_1, \phi_2, \ldots, \phi_k : V\Gamma \to V\Gamma$ as in the proof of Proposition 3.16. Then we have $d^{\rho'} = \sum_{i=1}^k d^{\phi_i \circ \rho'}/k$. Therefore there is an index i such that $\langle c, d^{\phi_i \circ \rho'} \rangle < \langle c, d^{\rho} \rangle$. We may assume that $1 \leq i \leq k/2$ by reversing the orientation if necessarily. Compare $\phi_i \circ \rho'$ with ρ. By the condition (4.2), if $(\phi_i \circ \rho')(x) \neq \rho(x)$, then

(4.3) (i) there is an oriented edge $\overrightarrow{pq} \in \overrightarrow{E\Gamma}$ such that $\rho(x) = p$ and $(\phi_i \circ \rho')(x) = q$, or

(ii) there are two oriented edges $\overrightarrow{pq}, \overrightarrow{qr} \in \overrightarrow{E\Gamma}$ belonging to a common 2-cell in Δ such that $\rho(x) = p$ and $(\phi_i \circ \rho')(x) = r$.

Indeed, we can take a cell σ in Δ containing both $\rho'(x)$ and $\rho(x)$. Then $(\phi_i \circ \rho')(x) \neq \rho(x)$ if and only if a shortest path P in Γ^k connecting $\rho'(x)$ and $\rho(x)$ crosses U_i; recall Figure 9. If $i > k/2$, then P never crosses U_i by (4.2). Thus if P crosses U_i, then $p = \rho(x)$ is necessarily a source of an oriented leg of σ. Since P crosses edges in U_i at most twice, the position of $(\phi_i \circ \rho')(x)$ is (i) (P crosses U_i once) or (ii) (P crosses U_i twice).

Let $\mathcal{O} = \{O_1, O_2, \ldots, O_m\}$ be the set of orbits of Δ. As in Section 3.4, we define maps $\varphi_j : V\Gamma \to V\Gamma^{O_j}$ by $p \mapsto p^{O_j}$ ($j = 1, 2, \ldots, m$). By Proposition 3.15, we have $d^{\rho} = \sum_{j=1}^k d^{\varphi_j \circ \rho}$ (for any potential ρ). Therefore we have $\sum_{j=1}^m \langle c, d^{\varphi_j \circ \phi_i \circ \rho'} \rangle < \sum_{j=1}^m \langle c, d^{\varphi_j \circ \rho} \rangle$. Then there is an index j with $\langle c, d^{\varphi_j \circ \phi_i \circ \rho'} \rangle < \langle c, d^{\varphi_j \circ \rho} \rangle$. Let $\overrightarrow{O_j}$ be the admissible orientation of O_j induced by $\overrightarrow{\Gamma}$. By using $\overrightarrow{O_j}$ and $\phi_i \circ \rho'$, we can construct a neighbor ρ'' of ρ with respect to $\overrightarrow{O_j}$ with $\langle c, d^{\rho''} \rangle < \langle c, d^{\rho} \rangle$ as follows. For each $x \in VG$, if there is $\overrightarrow{pq} \in \overrightarrow{O_j}$ with $\rho(x) = p$ and $(\varphi_j \circ \phi_i \circ \rho')(x) = q^{O_j}$ or there are $\overrightarrow{pr}, \overrightarrow{rq} \in \overrightarrow{O_j}$ belonging to a common 2-cell with $\rho(x) = p$ and $(\varphi_j \circ \phi_i \circ \rho')(x) = q^{O_j}$, then define $\rho''(x) = q$. For other vertex x, define $\rho''(x) = \rho(x)$. By construction, ρ'' is a neighbor of ρ with respect to $\overrightarrow{O_j}$, and we have

$$\varphi_l \circ \rho'' = \begin{cases} \varphi_j \circ \phi_i \circ \rho' & \text{if } l = j, \\ \varphi_l \circ \rho & \text{otherwise,} \end{cases} \qquad (l = 1, 2, \ldots, m).$$

Then we have $\langle c, d^{\rho''} \rangle = \sum_{l=1}^m \langle c, d^{\varphi_l \circ \rho''} \rangle = \langle c, d^{\varphi_j \circ \phi_i \circ \rho'} \rangle + \sum_{l \neq j} \langle c, d^{\varphi_l \circ \rho} \rangle < \sum_{l=1}^m \langle c, d^{\varphi_l \circ \rho} \rangle = \langle c, d^{\rho} \rangle$. Thus ρ'' is a required neighbor.

4.2 Distances of combinatorial dimension 2

For a distance μ on a finite set S, define two polyhedral sets P_μ and T_μ in $\mathbf{R}^S$ by

$$\begin{aligned} P_\mu &= \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \quad (s, t \in S)\}, \\ T_\mu &= \text{the set of minimal elements of } P_\mu. \end{aligned}$$

T_μ is called the *tight span* of μ [6, 15]. Its dimension $\dim T_\mu$ is defined to be the largest dimension of its faces, and is called the *combinatorial dimension* of μ in [6]. The geometry of T_μ reflects the combinatorial property of μ as follows:

(4.4) (i) For a metric μ, $\dim T_\mu \leq 1$ if and only if μ is a tree metric [6].

(ii) For a distance μ, $\dim T_\mu \leq 1$ if and only if μ is a distance between subtrees of a tree [10].

(iii) For a rational metric, $\dim T_\mu \leq 2$ if and only if μ is a dilation of a submetric of a frame [18].

The main aim here is to give an extension of (iii) to a nonmetric version as follows.

Theorem 4.3. *Let μ be a rational distance on a finite set S. The following two conditions are equivalent:*

(1) $\dim T_\mu \leq 2$.

(2) *μ has an F-complex realization.*

The larger part of the proof has already been given in [11]. So we sketch it. First we show (2) $\Rightarrow$ (1). Suppose that μ has an F-complex realization. Then by Theorem 2.1 there exists a positive integer k (the denominator of rational leg-length κ) such that LP-dual (3.2) always has a $1/k$-integral optimal solution. Namely, the *dual fractionality* of μ is finite. Then [11, Theorem 1.1] implies $\dim T_\mu \leq 2$.

Second, we show (1) $\Rightarrow$ (2). Suppose $\dim T_\mu \leq 2$. Then [11] showed that T_μ has a polyhedral subdivision $\varDelta$ with property (2.1), and (T_μ, l_∞) is isometric to $(|\varDelta|, d_\varDelta)$. For $s \in S$, define a subset $T_{\mu,s}$ by $\{p \in T_\mu \mid p(s) = 0\}$. We verify that $(\varDelta; \{T_{\mu,s}\}_{s \in S})$ is an F-complex realization of μ. It is known that $d_\varDelta(T_{\mu,s}, T_{\mu,t}) = \mu(s,t)$ [11, Lemma 3.6]. Hence it suffices to verify that $\varDelta$ is an F-complex and each $T_{\mu,s}$ is normal. Although one can directly prove that $\varDelta$ satisfies (2.3), this approach needs full details of [11, Section 3]. Instead, we use Chepoi's characterization [3] of F-complexes in terms of hyperconvexity. Here a metric space is said to be *hyperconvex* (in the sense of [1]) if the family of balls (around points) has the Helly property.

Theorem 4.4 ([3, Theorem 7.8])**.** *A locally-finite and simply-connected 2-complex $\varDelta$ satisfying (2.1) without triangle-folders is an F-complex if and only if $(|\varDelta|, d_\varDelta)$ is hyperconvex.*

In fact, this theorem holds for any finite and simply-connected 2-complex with property (2.1) possibly having triangle-folders. Indeed, the only if-part follows from Theorem 3.13 (with some care for infinite families and irrational radius). Also the if-part follows from a slight change in Chepoi's proof by the following way. Suppose that $(|\varDelta|, d_\varDelta)$ is hyperconvex, and suppose to the contrary that there are three folders F_1, F_2, F_3 violating (2.3) at a common point p. Some of F_1, F_2, F_3 may be triangle-folders. In this case, consider 2-subdivision $\varDelta^2$. Again we can take folders violating (2.3) at p each of which is not a triangle-folder. Then apply the argument in [3, p. 156] to them, which leads a contradiction. So it suffices to verify the following, which is well-known in the literature for metrics μ.

Lemma 4.5. *(T_μ, l_∞) is hyperconvex.*

This is an easy consequence of the next lemma.

Lemma 4.6 ([6, p.331, (1.9)])**.** *There exists a map $\phi : P_\mu \to T_\mu$ such that*

(1) $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ *for $p, q \in P_\mu$, and*

(2) $\phi(p) \leq p$ *for $p \in P_\mu$, and thus ϕ is identical on T_μ.*

(This lemma holds for nonmetric distances μ although it was originally stated for metrics.)

The proof of Lemma 4.5. For $\{p_i\}_{i \in I} \subseteq T_\mu$ and $\{r_i\}_{i \in I} \subseteq \mathbf{R}_+$, suppose that the collection of balls $\{B(p_i, r_i)\}_{i \in I}$ in $(\mathbf{R}^S, l_\infty)$ has a nonempty pairwise intersection at T_μ. Each ball $B(p_i, r_i) \subseteq \mathbf{R}^S$ is a direct product of segments $[p_i(s) - r_i, p_i(s) + r_i]$ and thus $\{B(p_i, r_i)\}_{i \in I}$ has the Helly property on $(\mathbf{R}^S, l_\infty)$. Let p^* be a unique maximal element in $\bigcap_{i \in I} B(p_i, r_i)$, which is nonempty. Then one can verify that p^* belongs to P_μ. Take a nonexpansive map ϕ from the previous lemma. Then $\phi(p^*)$ belongs to $\bigcap_{i \in I} T_\mu \cap B(p_i, r_i)$. □

Finally we verify that $T_{\mu,s}$ is normal. For a face F of P_μ, let K_F be the graph on S with edge set $EK_F = \{st \mid s, t \in S,\ p(s) + p(t) = \mu(s,t)\ (\forall p \in F)\}$. Namely K_F represents the set of facets of P_μ active at F. Note that K_F has a loop ss exactly when $F \subseteq \{p \in \mathbf{R}^S \mid p(s) = 0\}$. Then F belongs to T_μ if and only if K_F has no isolated vertices. $T_{\mu,s}$ is the union of subcomplex of Δ since $T_{\mu,s}$ is the union of faces F of T_μ whose K_F has a loop at s. The connectivity is obvious. We show that every face containing e belongs to $T_{\mu,s}$, which implies (2.4) (i). A leg coincides with an l_1-*edge* in the sense of [11]. Let F be a minimal face of T_μ containing e. In the graph K_F, s belongs to the connected component each of whose vertices has a loop [11, Lemma 3.8]. So every face F' of T_μ containing F necessarily belongs to $T_{\mu,s}$. Indeed, by $EK_{F'} \subseteq EK_F$, $ss \notin EK_{F'}$ implies that s is isolated in $K_{F'}$ ($F' \not\subseteq T_\mu$). Suppose to the contrary that $T_{\mu,s}$ violates the local convexity condition, i.e., there are two triangles $\sigma, \sigma' \in \Delta$ with vertices p, r, v and q, r, v (resp.) such that pr and rq are hypotenuses with $T_{\mu,s} \cap (\sigma \cup \sigma') = pr \cap rq$. Then $\|p - q\|_\infty = 2\kappa$. Consider a geodesic P connecting p and r in $\{p \in P_\mu \mid p(s) = 0\}$. Consider the image $\phi(P)$ of P by a nonexpansive map ϕ in Lemma 4.6. Then $\phi(P)$ is also a geodesic connecting p and q in $T_{\mu,s}$ with length 2κ. By (3.6) and (2.4) (i), Δ must have a triangle with vertices p, q, v and hypotenuse pq, which violates the flag condition at v. A contradiction.

Remark 4.7. A metric space X is called *injective* (or *an absolute retract*) if for every metric space Y containing X as a subspace there exists a nonexpansive retraction from Y to X. Aronszajn and Panitchpakdi [1] showed that X is injective if and only if X is hyperconvex. From the point of the view, two Lemmas 4.5 and 4.6 are essentially equivalent.

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