Multiflows and Metrics

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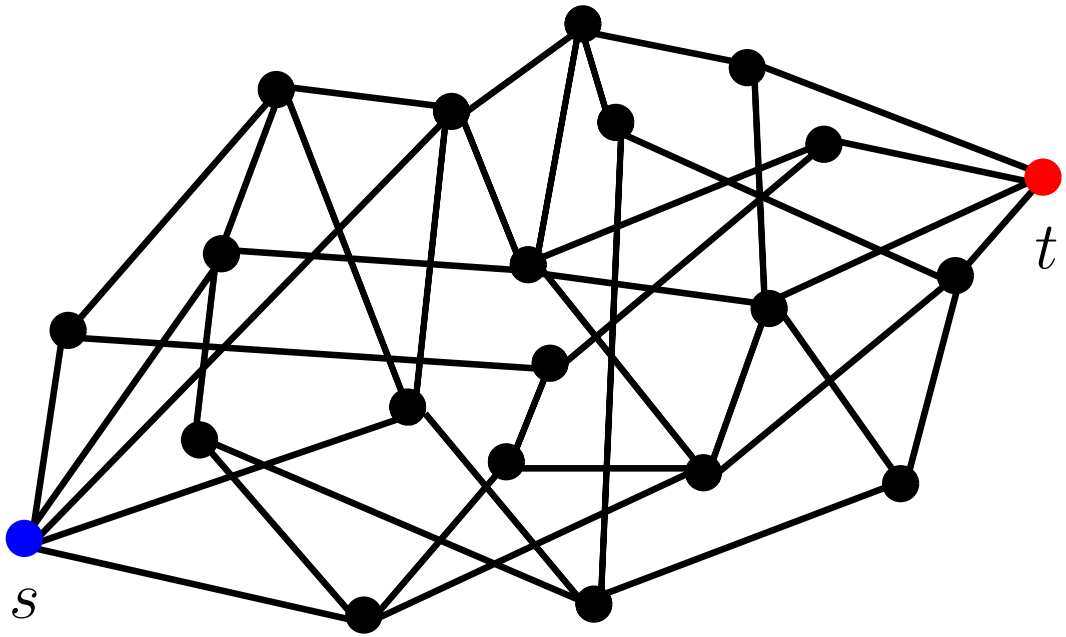
Part I: Flows

Network flows (single commodity flows)

$G = (V, E)$: an undirected graph
$c : E \to \mathbf{R}_+$: an edge capasity
$s, t \in V$: terminals

Def: s-t flow $f = (\mathcal{P}, \lambda)$
$\mathcal{P}$: a set of s-t paths
$\lambda : \mathcal{P} \to \mathbf{R}_+$: a flow-value function satisfying capasity constraint

$$\sum\{\lambda(P) \mid P \in \mathcal{P}, e \in P\} \le c(e) \quad (e \in E).$$

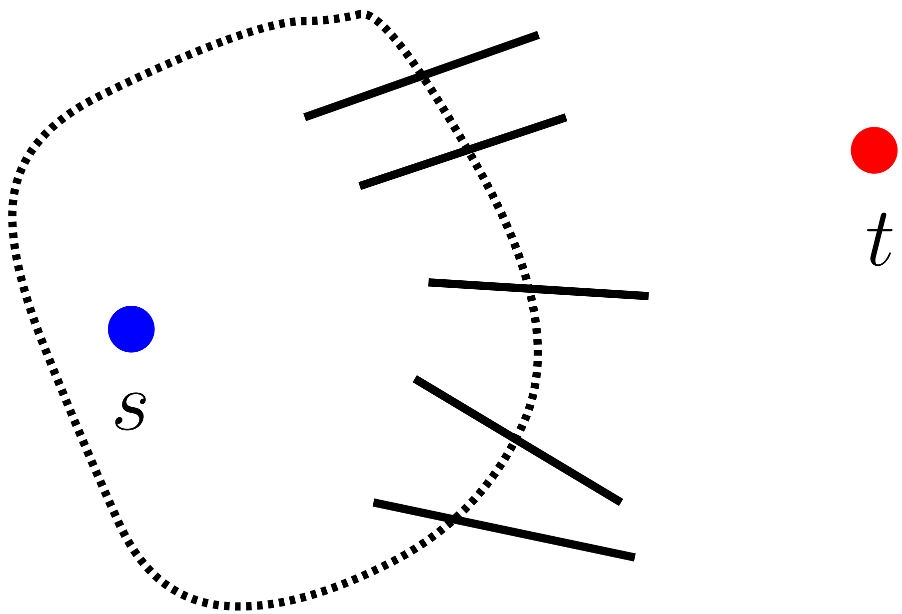


Maximum flow problem:

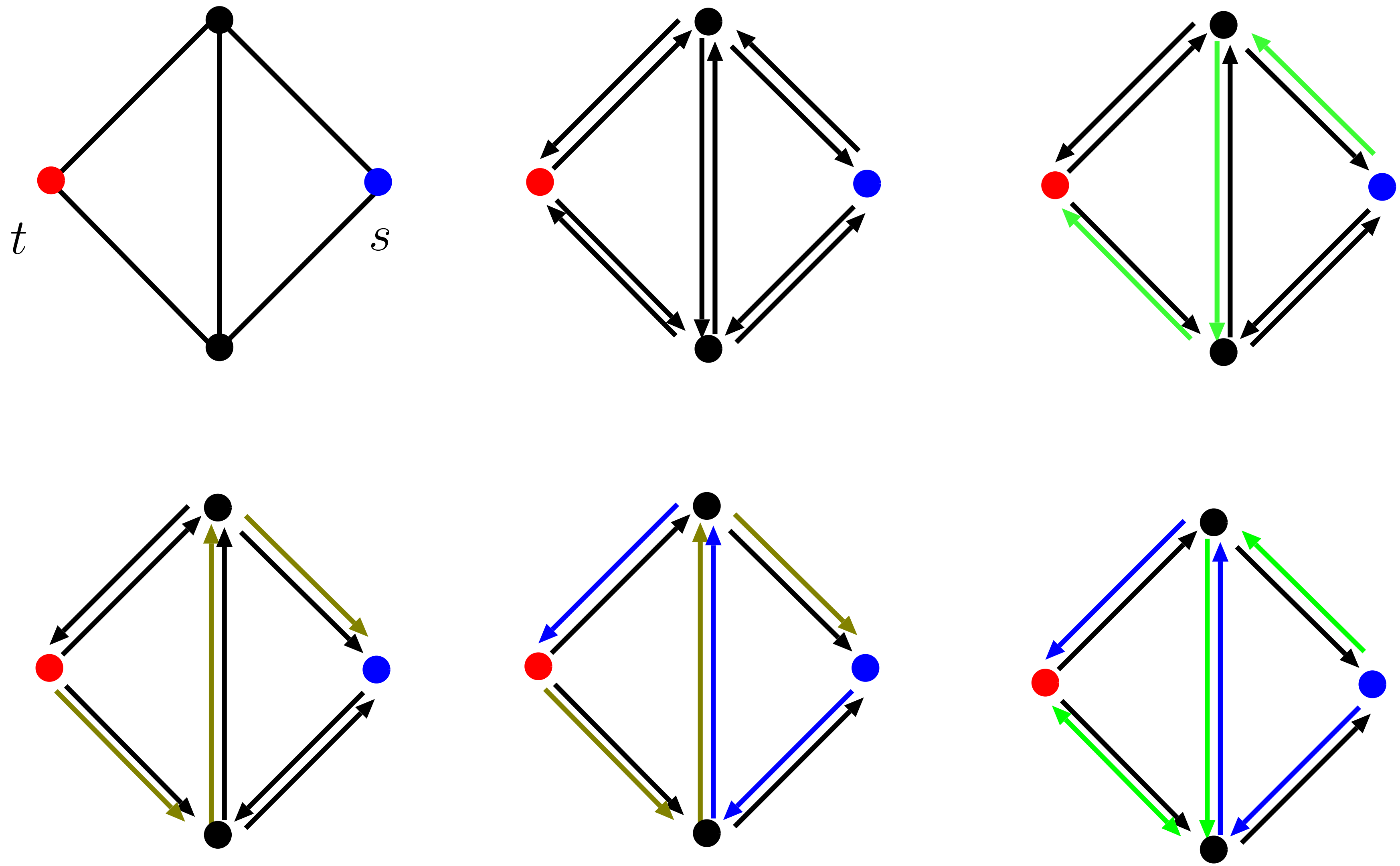
Maximize $\sum_{P\in\mathcal{P}}\lambda(P)$ over s-t flows $f=(\mathcal{P},\lambda)$.

Maxflow-Mincut Theorem (Ford-Fulkerson 56)

- Maximum s-t flow value = Minimum s-t cut value,
- $\exists$ integral optimal flow if c is integral.



Sketch of the proof (augmenting path):

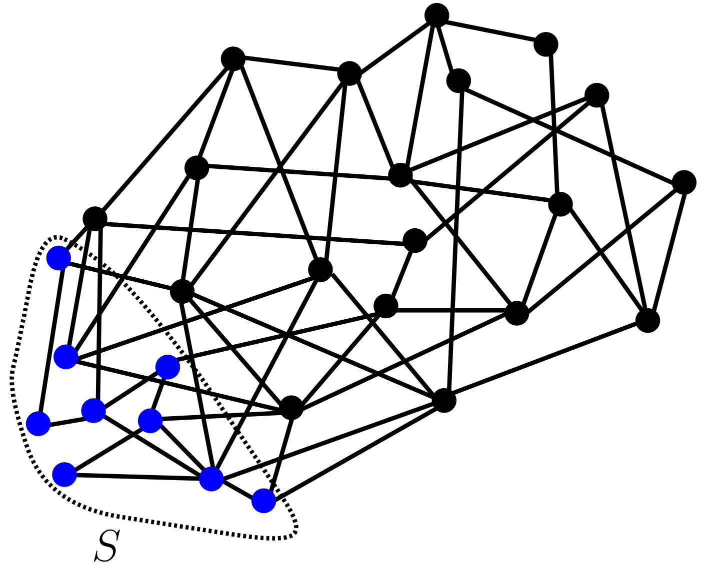


Multicommodity flows (multiflows)

$G = (V, E)$: an undirected graph
$c : E \to \mathbf{R}_+$: an edge capasity
$S \subseteq V$: a set of terminals

Def: multiflow $f = (\mathcal{P}, \lambda)$ for $(G, c; S)$
$\mathcal{P}$: a set of S-paths
$\lambda : \mathcal{P} \to \mathbf{R}_+$: a flow-value function satisfying capasity constraint

$$\sum\{\lambda(P) \mid P \in \mathcal{P}, e \in P\} \leq c(e) \quad (e \in E).$$



μ-weighted maximum multiflow problem:

Given $\mu : S \times S \to \mathbf{R}_+$ with $\mu(s,t) = \mu(t,s) \geq \mu(s,s) = 0$,

$$\begin{array}{rl} \text{Maximize} & \displaystyle\sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \\ \text{Subject to} & f = (\mathcal{P}, \lambda) : \text{a multiflow for } (G, c; S), \end{array}$$

where s_P, t_P: endpoints of P.

Examples:

- Two-commodity flow maximization:
  $S = \{s, s', t, t'\}$, $\mu(s,t) = \mu(s',t') = 1$ and zero otherwise.

- Free multiflow problem:
  $\mu(s,t) = 1$ for $s \neq t$

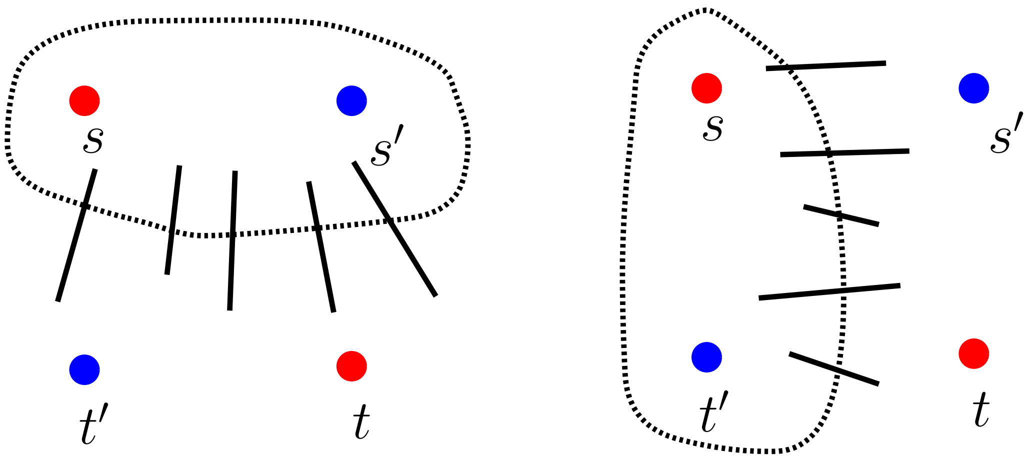
Rem: 0-1 weight $\mu \Leftrightarrow$ commodity graph H.

an integral optimal flow may not exist even if c is integral !

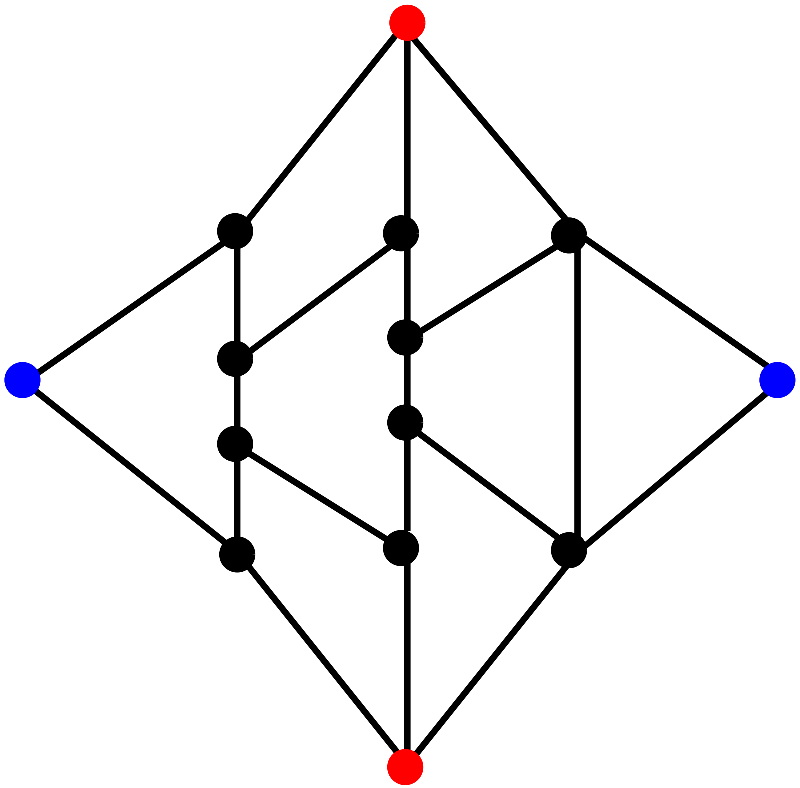
Two-commodity flows

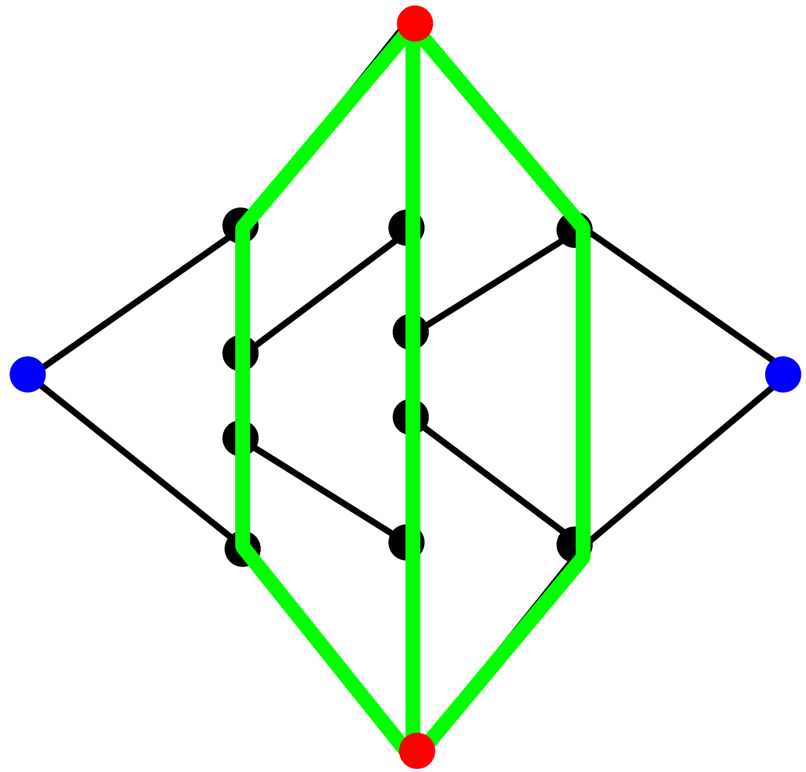
Maxbiflow-Mincut Theorem (Hu 63)

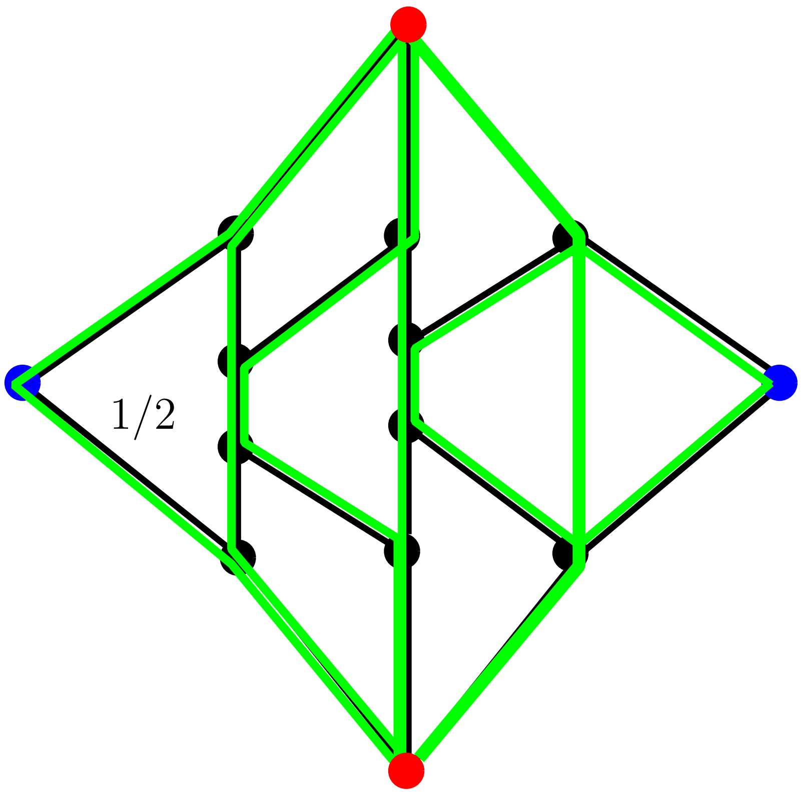
- Maximum flow value = Min (ss'-tt' mincut, st'-ts' mincut)
- $\exists$ half-integral optimal multiflow if c is integral.



Sketch of the proof (Hu's original proof: double-path)



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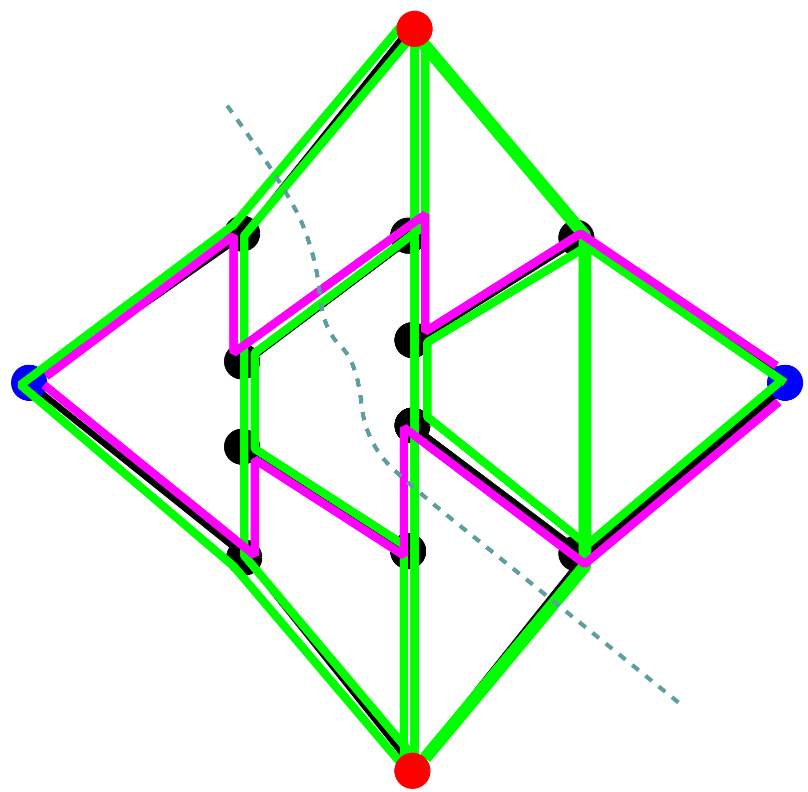
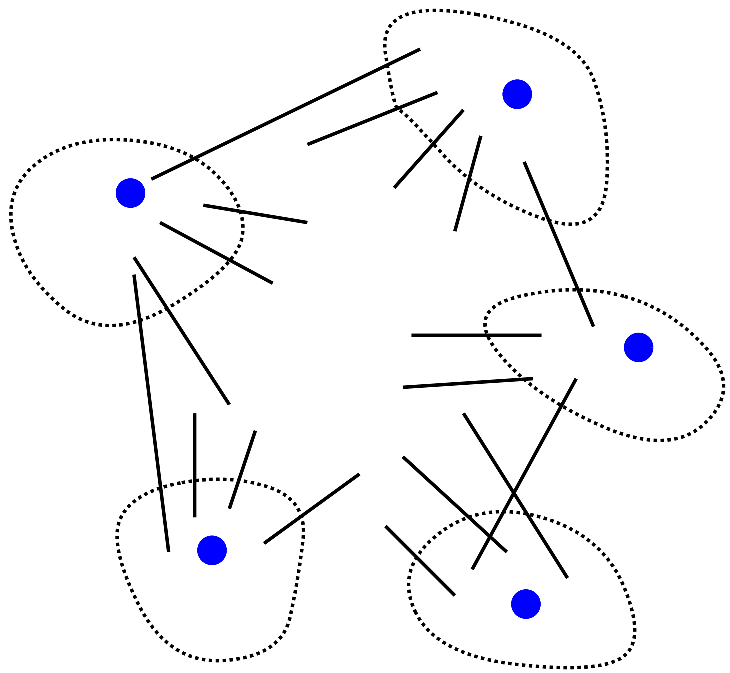


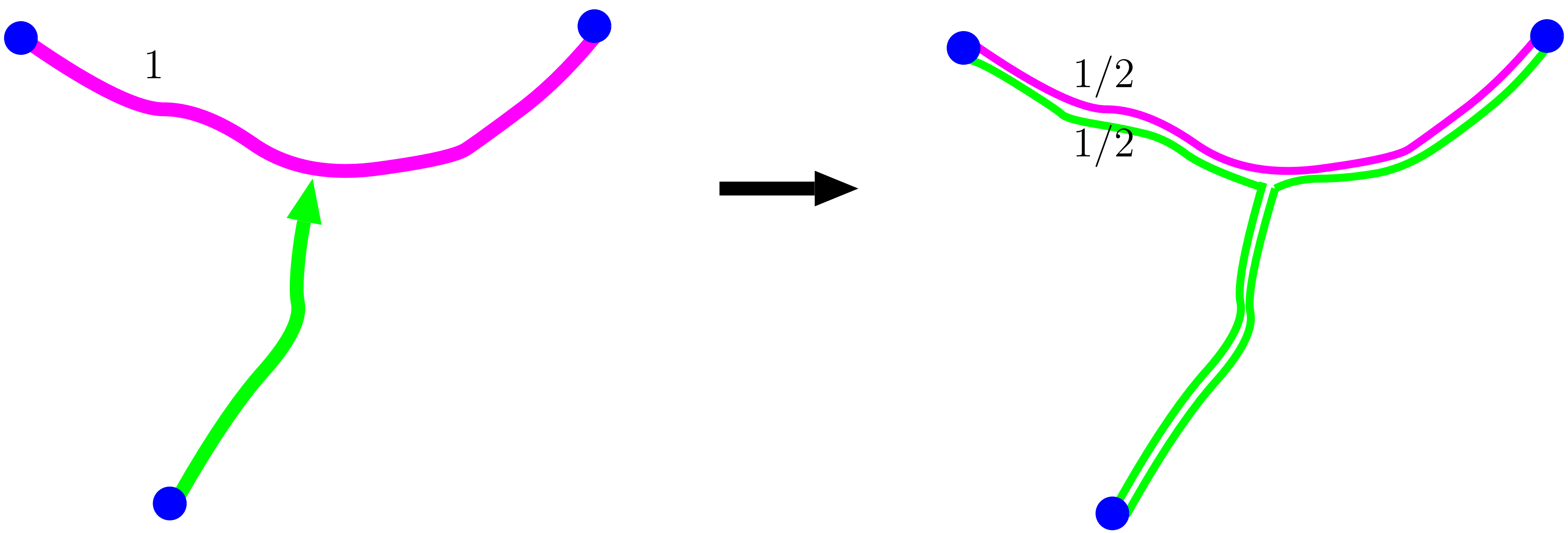
Free multiflows

Theorem (Lovász 76, Cherkassky 77)

- Maximum flow value $= \frac{1}{2}\sum_{t\in S} t\text{-}S\setminus t$ mincut.
- $\exists$ half-integral optimal multiflow if c is integral.



Sketch of the proof (Cherkassky's T-operation)



Remarks

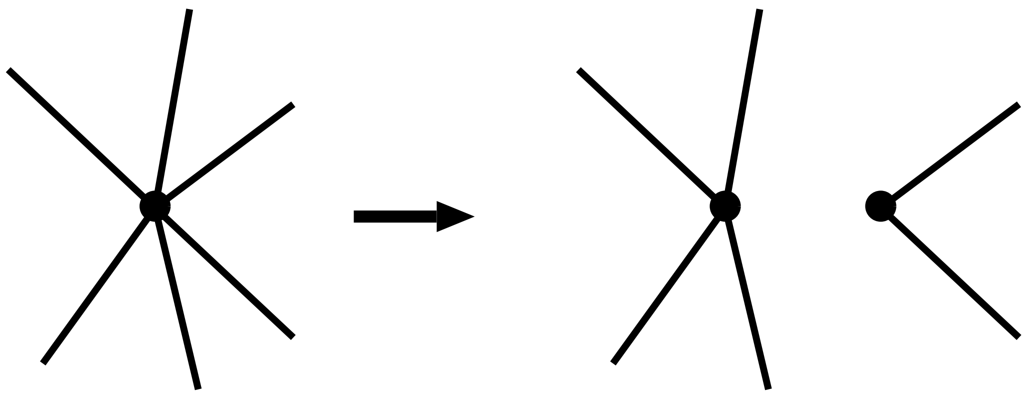
- A common generalization of two theorems above

⇒ Karzanov-Lomonosov Theorem (Karzanov-Lomonosov 78)

- Other proof method

⇒ Splitting-off (Rothschild-Winston 66, Lovász 76)

(G, c): Eulerian graph



Fractionality problem (Karzanov 89)

Def: The fractionality of weight μ

$\overset{\text{def}}{\Longleftrightarrow}$

the least positive integer k such that for every integer-capasitated graph, the μ-problem has a $1/k$-integer optimal multiflow.

Ex: 3-commodity weight $\mu \Rightarrow$ the fractionality is infinity.

Ex: (Karzanov-Manossakiss 96): $\mu = \mathrm{dist}_{K_{2,m}} \Rightarrow$ the fractionality is 2.

When is the fractionality finite ?

Karzanov Conjecture (Karzanov 90, ICM, Kyoto)

0-1 weight $\mu \Leftrightarrow$ commodity graph H

Theorem: (Karzanov 89)
If the fractionality of H is finite, then

(*) any triple A, B, C of maximal stable sets of H satisfies
$A \cap B = B \cap C = C \cap A$.

Conjecture: (Karzanov 90)
If H satisfies (*), then the fractionality is finite, and is one of $1, 2, 4$.

Part II: Metrics

Message:

LP-dual of multiflow

$\Rightarrow$ an optimization over metrics

$\Rightarrow$ a location problem on some polyhedral space (tight span)

$\Rightarrow$ a location problem on some graph

$\Rightarrow$ a combinatorial duality theorem

Linear programming duality

$$\begin{array}{lll} \text{max.} & \mu^\top x & \simeq \\ \text{s.t.} & Ax \le c & \\ & x \ge 0 & \end{array} \quad \begin{array}{ll} \text{min.} & y^\top c \\ \text{s.t.} & y^\top A \ge \mu \\ & y \ge 0 \end{array}$$

Lemma (Edmonds-Giles 77)

Suppose μ is integral,

$\forall c$ integral $\exists$ $1/k$-integral optimum x^*,

$\Rightarrow \{y \ge 0 : y^\top A \ge \mu\}$ is $1/k$-integral.

Multiflow maximization as a linear programming

$$
\begin{aligned}
\text{max.} \quad & \sum_{P \in \mathcal{P}} \mu(s_P, t_P)\lambda(P) \\
\text{s.t.} \quad & \sum \{\lambda(P) \mid P \in \mathcal{P}, e \in P\} \leq c(e) \quad (e \in E). \\
& \lambda \geq 0. \\
\simeq & \\
\text{min.} \quad & \sum_{e \in E} c(e) l(e) \\
\text{s.t.} \quad & \sum_{e \in P} l(e) \geq \mu(s_P, t_P) \quad (P \in \mathcal{P}). \\
& l(e) \geq 0.
\end{aligned}
$$

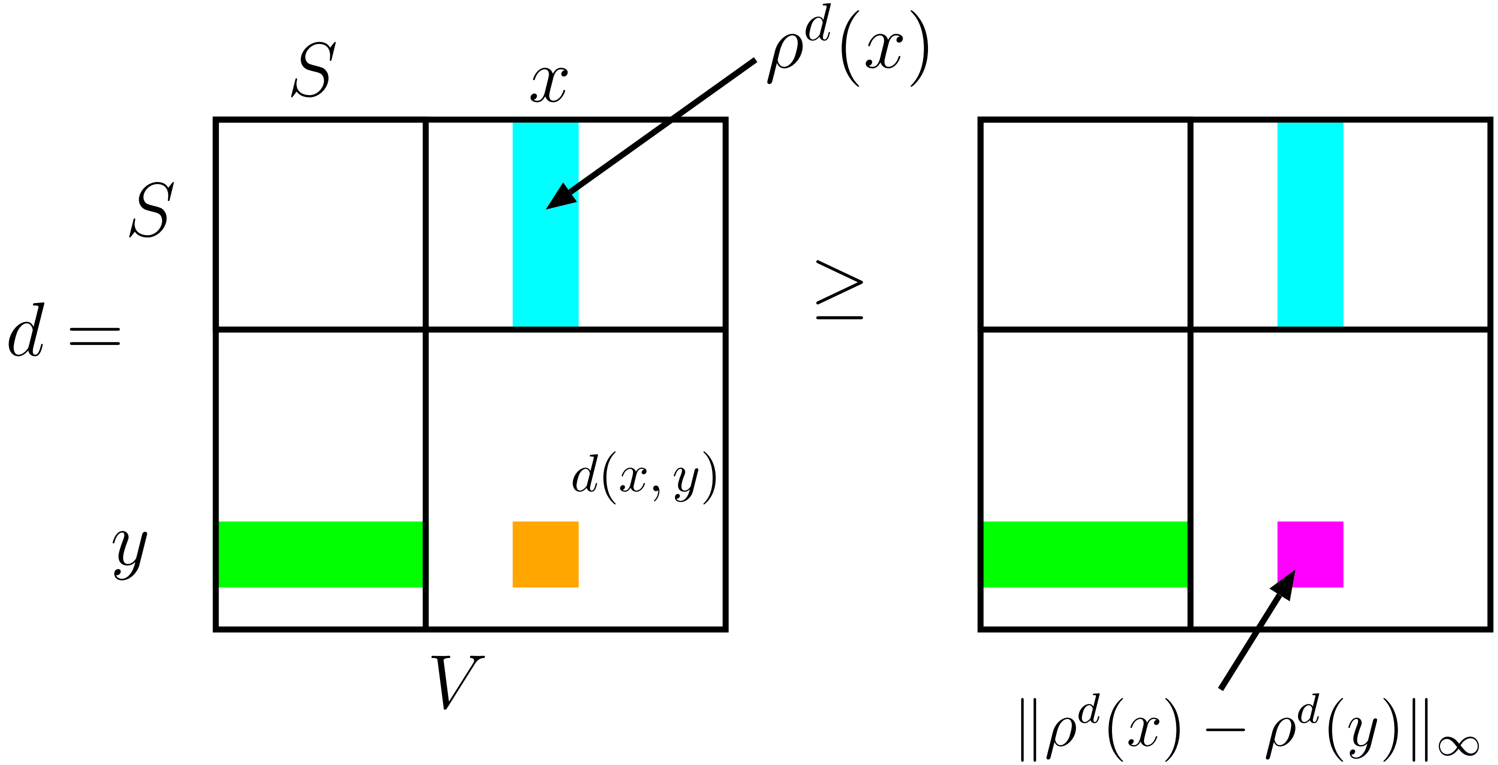
Duality between multiflows and metrics (Onaga-Kakusho, Iri 71)

$$
\begin{array}{rl}
\text{min.} & \displaystyle\sum_{e \in E} c(e) l(e) \\
\text{s.t.} & \displaystyle\sum_{e \in P} l(e) \geq \mu(s_P, t_P) \quad (e \in E), \\
 & l(e) \geq 0. \\
\simeq & \\
\text{min.} & \displaystyle\sum_{e \in E} c(e) d(e) \\
\text{s.t.} & d\text{: metric on } V, \\
 & d(s,t) \geq \mu(s,t) \quad (s, t \in S).
\end{array}
$$

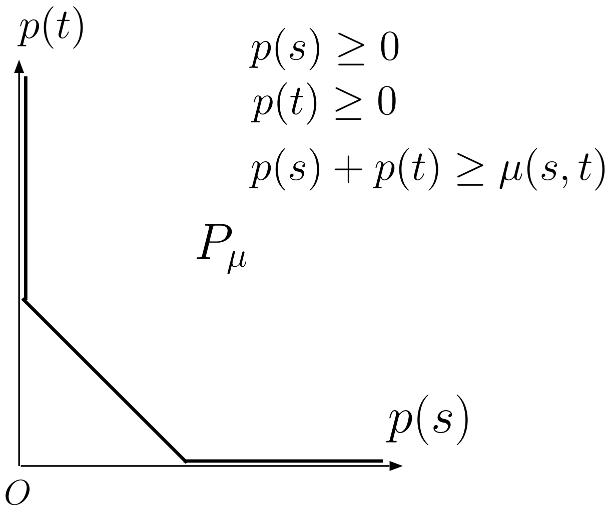
hint: $l(e) \geq \text{dist}_{G,l}(e)$.

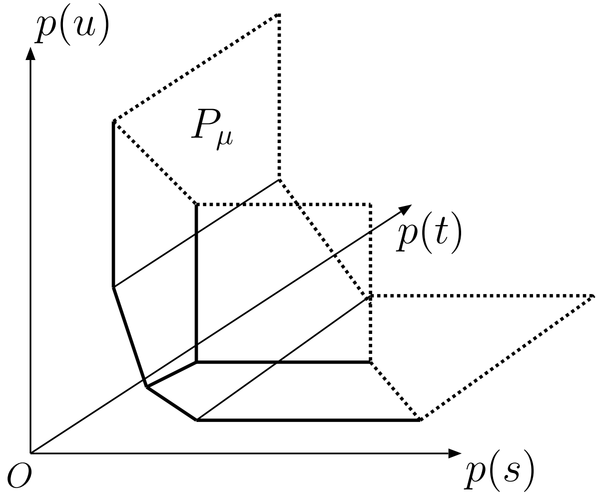
A key observation

$$\begin{array}{rl} \text{Min.} & \displaystyle\sum_{e \in E} c(e)d(e) \\ \text{s.t.} & d\text{: metric on } V, \\ & d(s,t) \geq \mu(s,t) \quad (s,t \in S). \end{array}$$



$$P_\mu = \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s,t \in S)\}$$
$$P_{\mu,s} = \{s \in P_\mu \mid p(s) = 0\} \ (s \in S)$$





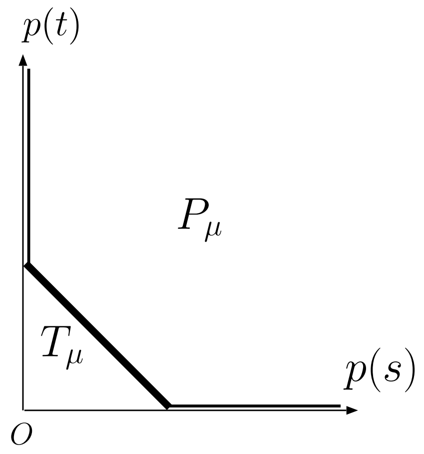
Lemma (H. 08): LP-dual of the μ-problem is equivalent to

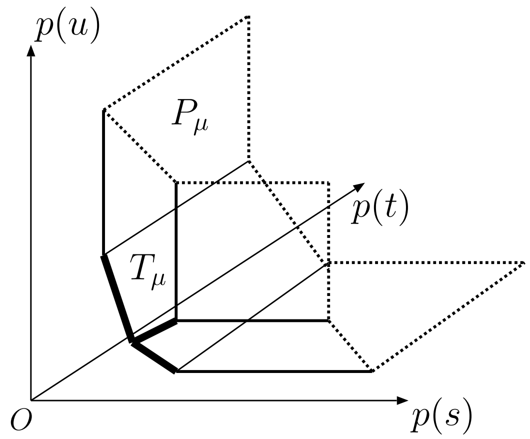
$$\begin{aligned} \text{Min.} \quad & \sum_{xy \in E} c(xy) \|\rho(x) - \rho(y)\|_\infty \\ \text{s.t.} \quad & \rho : V \to P_\mu \\ & \rho(s) \in P_{\mu,s} \quad (s \in S) \end{aligned}$$

$P_\mu := \{p \in \mathbf{R}^S \mid p(s) + p(t) \geq \mu(s,t) \ (s, t \in S)\}$
$T_\mu :=$ the set of minimal elements of P_μ ($\leftarrow$ tight span)
$T_{\mu,s} := \{p \in T_\mu \mid p(s) = 0\} \quad (s \in S)$

Lemma (Dress 84)
There is $\phi : P_\mu \to T_\mu$ such that

- $\phi(p) \leq p$ for $p \in P_\mu$ (, and thus $\phi(p) = p$ for $p \in T_\mu$),
- $\|\phi(p) - \phi(q)\|_\infty \leq \|p - q\|_\infty$ for $p, q \in P_\mu$.

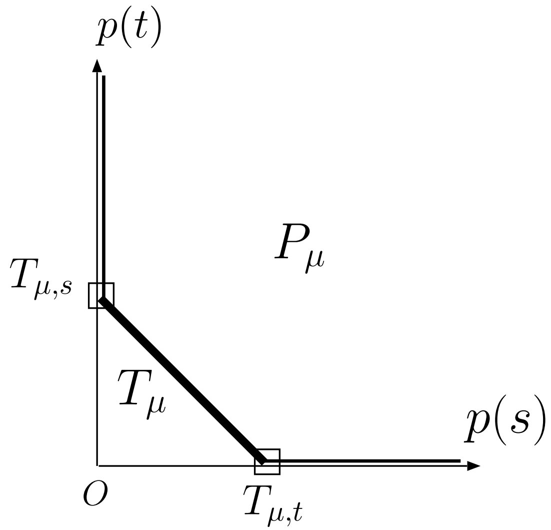


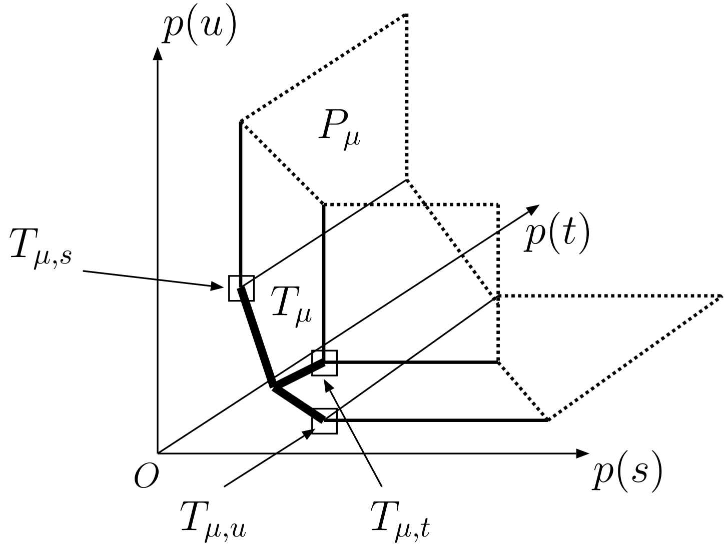


T-dual to the μ-problem:

Theorem (H. 07) LP-dual of the μ-problem is equivalent to

$$\begin{array}{ll} \text{Min.} & \displaystyle\sum_{xy\in E} c(xy)\|\rho(x)-\rho(y)\|_\infty \\ \text{s. t.} & \rho : V \to T_\mu \\ & \rho(s) \in T_{\mu,s} \quad (s \in S) \end{array}$$

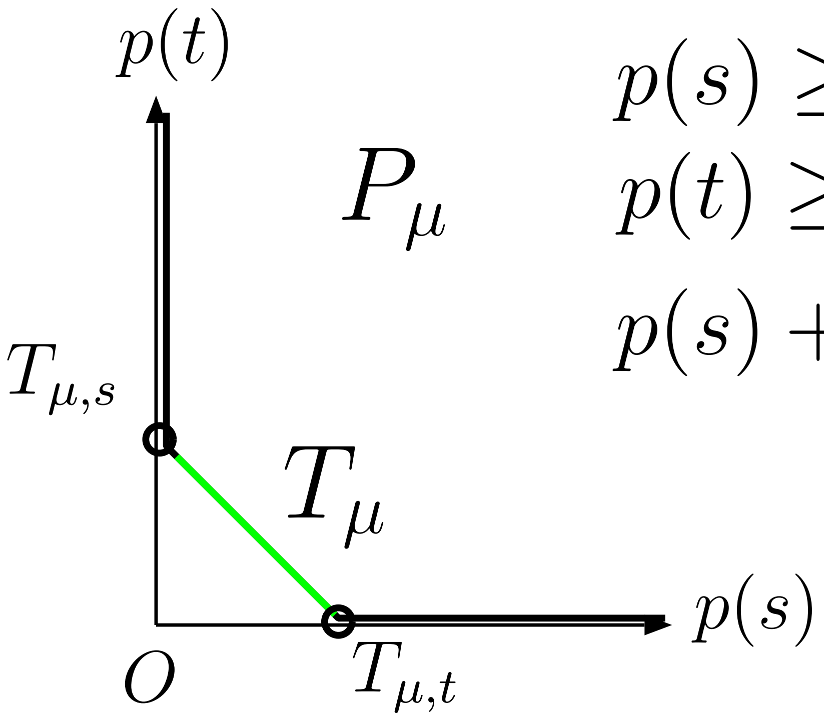




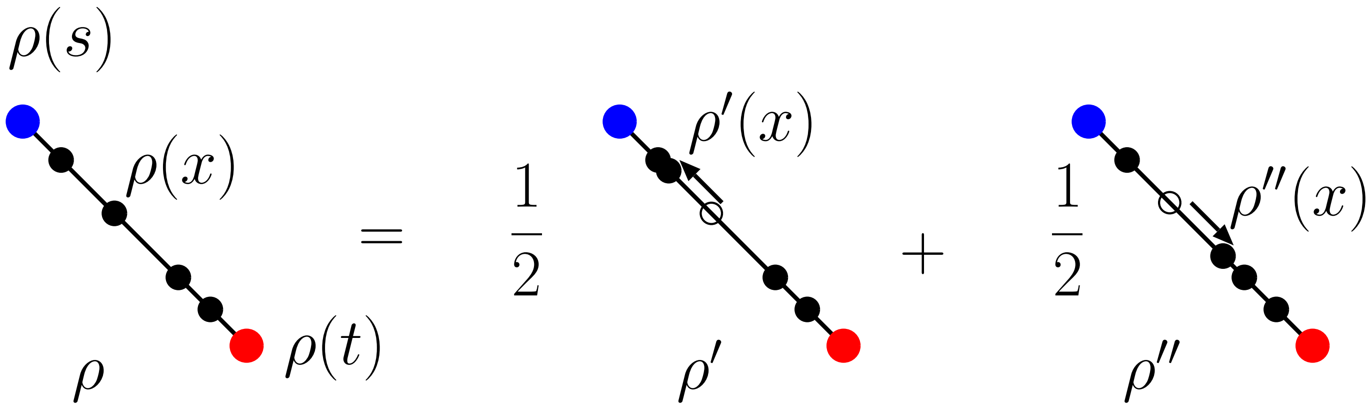
Interetation ?

Ford-Fulkerson reconsidered ($S = \{s, t\}$)

The tight span is a segment

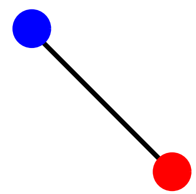


$$p(s) \geq 0$$
$$p(t) \geq 0$$
$$p(s) + p(t) \geq 1$$



$$
\begin{aligned}
d^{\rho} &= 1/2(d^{\rho'} + d^{\rho''}) \\
d^{\rho}(x,y) &:= \|\rho(x) - \rho(y)\|_{\infty}
\end{aligned}
$$

$\Rightarrow$ T-dual is equivalent to

Minimize $\sum_{xy \in E} c(xy)\,\mathrm{dist}(\rho(x), \rho(y))$ Subject to $\rho : V \to$ 

$\rho(s) =$ $\quad$ $\rho(t) =$

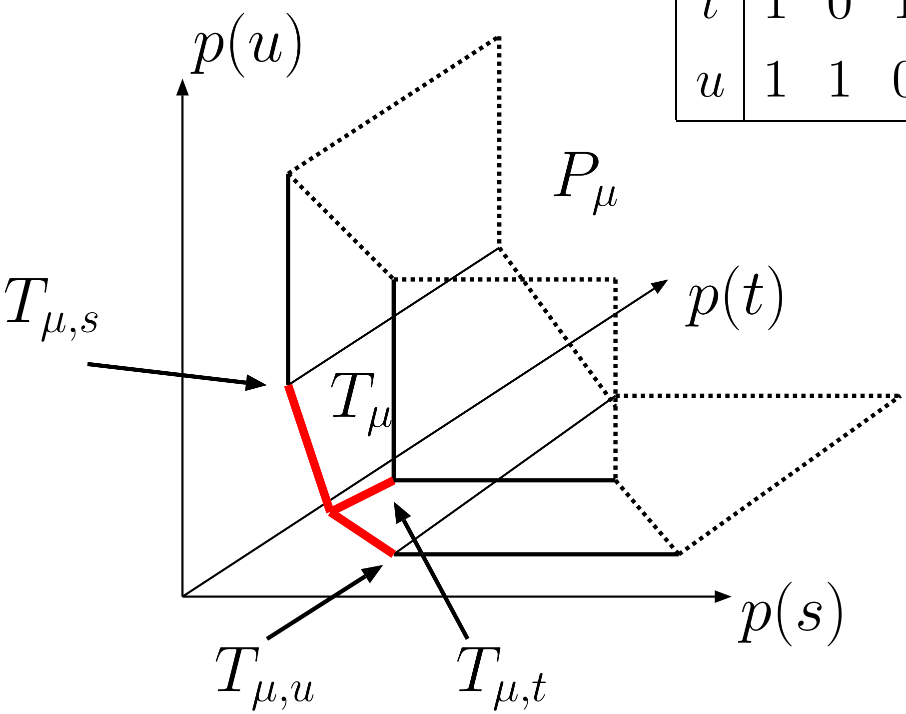
$\Rightarrow$ finding s-t mincut.

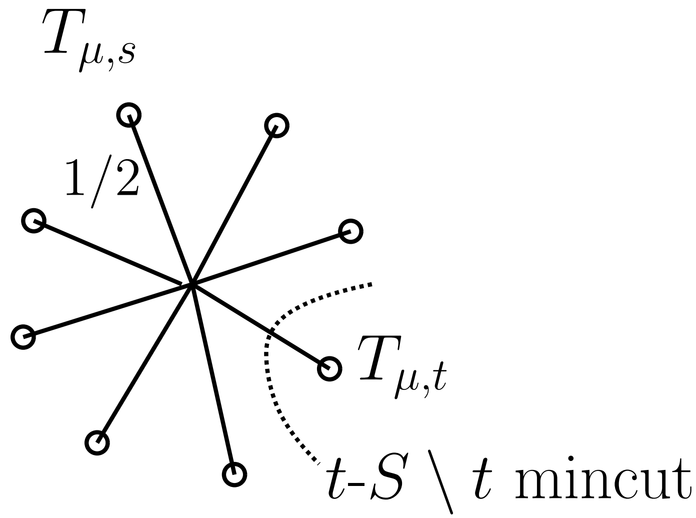
Lovász-Cherkassky reconsidered ($H_\mu = K_n$)

The tight span is a star

$S = \{s, t, u\}$

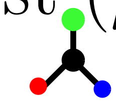
$$\mu = \begin{array}{|c|ccc|}\hline & s & t & u \\ \hline s & 0 & 1 & 1 \\ t & 1 & 0 & 1 \\ u & 1 & 1 & 0 \\ \hline \end{array}$$



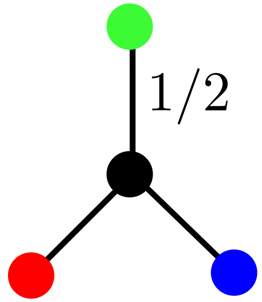


$\Rightarrow$ T-dual is equivalent to

$$\text{Minimize} \quad \sum_{xy \in E} c(xy)\,\text{dist}\,(\rho(x), \rho(y))$$



Subject to

$\rho : V \rightarrow$ 

$\rho(s) =$ 

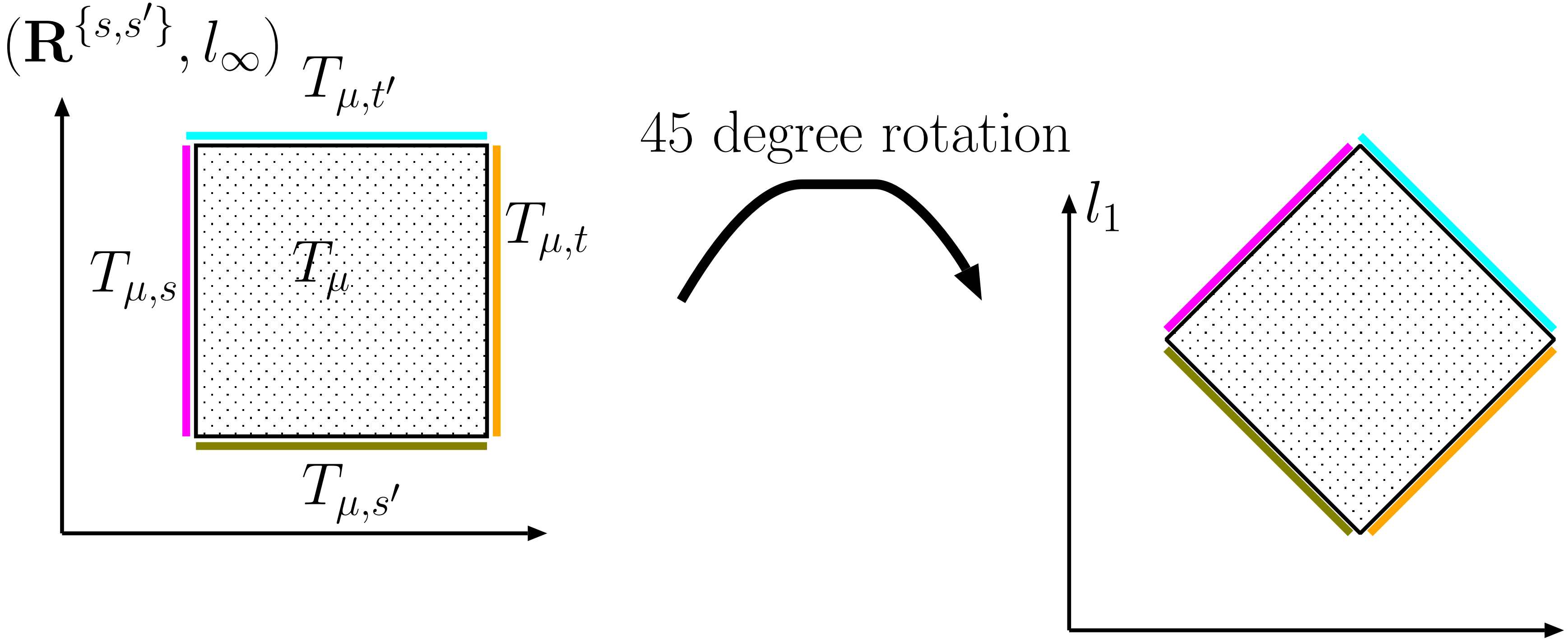
$\rho(t) =$ 

$\rho(u) =$ 

$\Rightarrow \dfrac{1}{2} \sum_{t \in S} t\text{-}S \setminus t$ mincut.

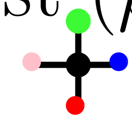
Two-commodity reconsidered ($S = \{s, t, s', t'\}$)

The tight span is a square in l_∞-plane ($\simeq$ l_1-plane).

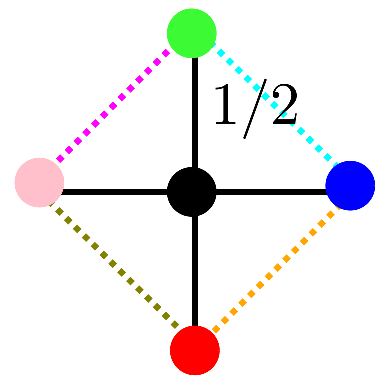


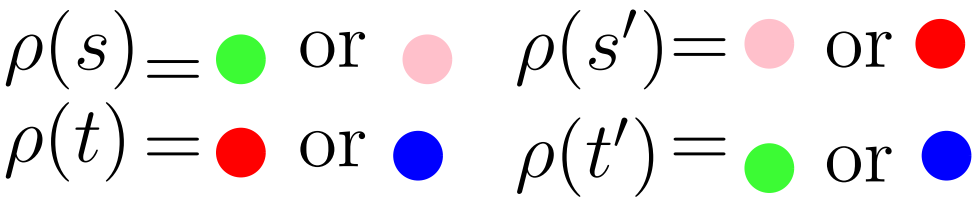
Rem: $(x_1, x_2) \mapsto \left(\frac{x_1 + x_2}{2}, \frac{x_1 - x_2}{2}\right)$.

T-dual is equivalent to

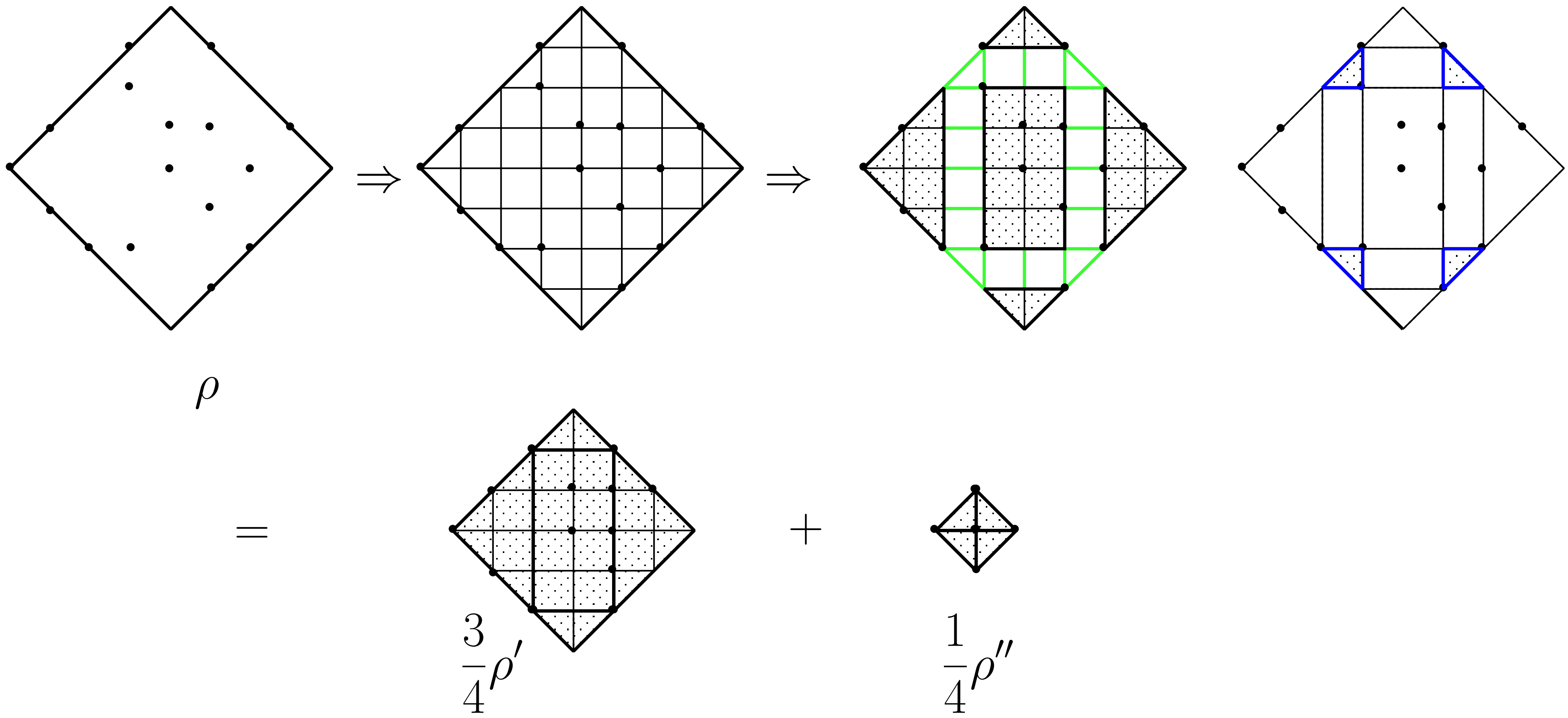
Minimize $\sum_{xy \in E} c(xy) \mathrm{dist}$  $(\rho(x), \rho(y))$

Subject to

$\rho : V \rightarrow$ 



Sketch of the proof



$$d^{\rho} = \frac{3}{4} d^{\rho'} + \frac{1}{4} d^{\rho''}$$

Theorem (Karzanov 98 for metrics, H. 07 for general)

• If $\dim T_\mu \leq 2$, then there exist

a graph Γ, vertices $L_s \subseteq V\Gamma$ for each $s \in S$, a positive integer k

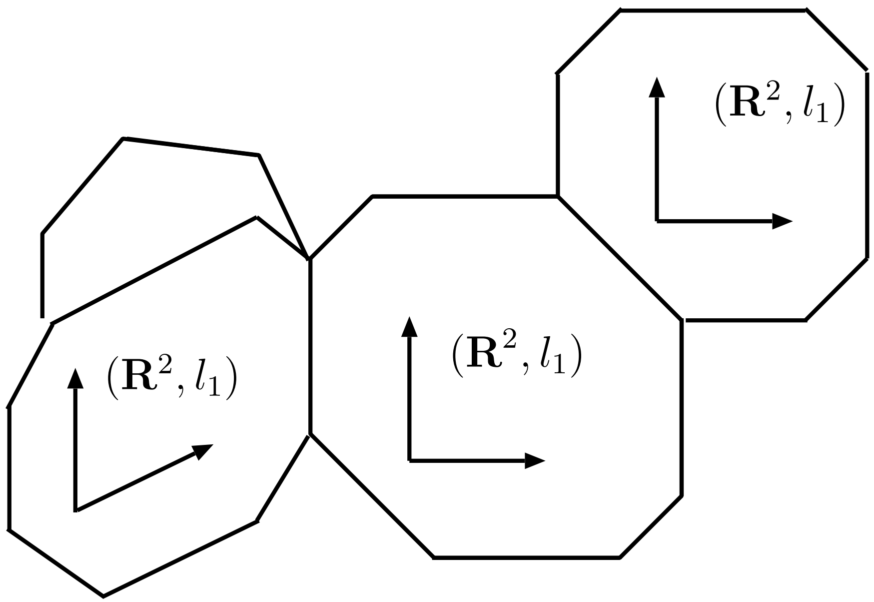
such that LP-dual of μ-problem is equivent to

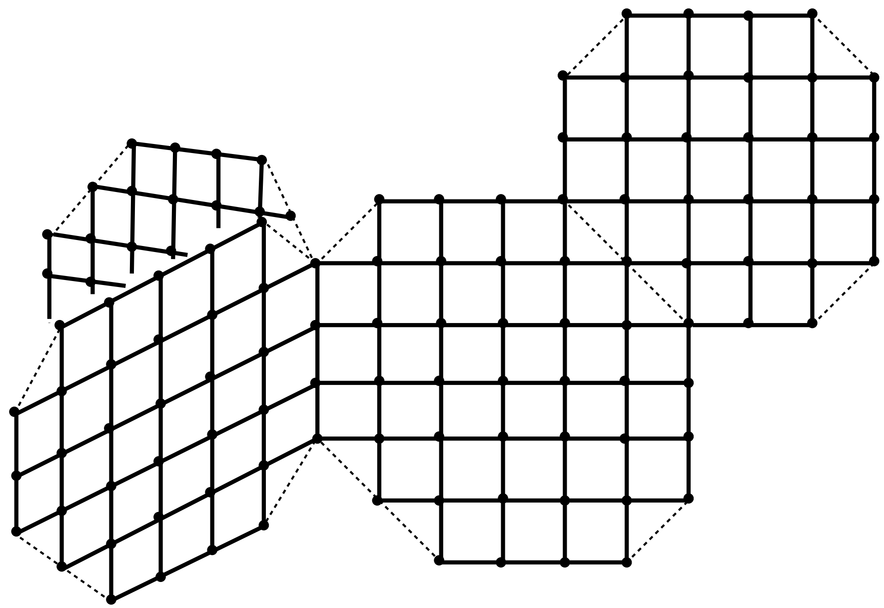
$$
\begin{array}{rl}
\text{Min.} & \displaystyle \frac{1}{k} \sum_{xy \in E} c(xy)\,\mathrm{dist}_\Gamma(\rho(x), \rho(y)) \\
\text{s.t.} & \rho : V \to V\Gamma, \\
 & \rho(s) \in L_s \quad (s \in S).
\end{array}
$$

• In addition, if μ is integral, $k \in \{1, 2, 4\}$.

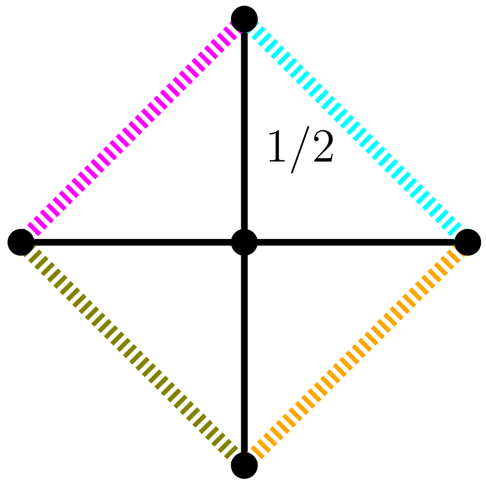
• If $\dim T_\mu > 2$, then there is no such a graph, and the fractionality of μ is infinity.

Sketch of the proof

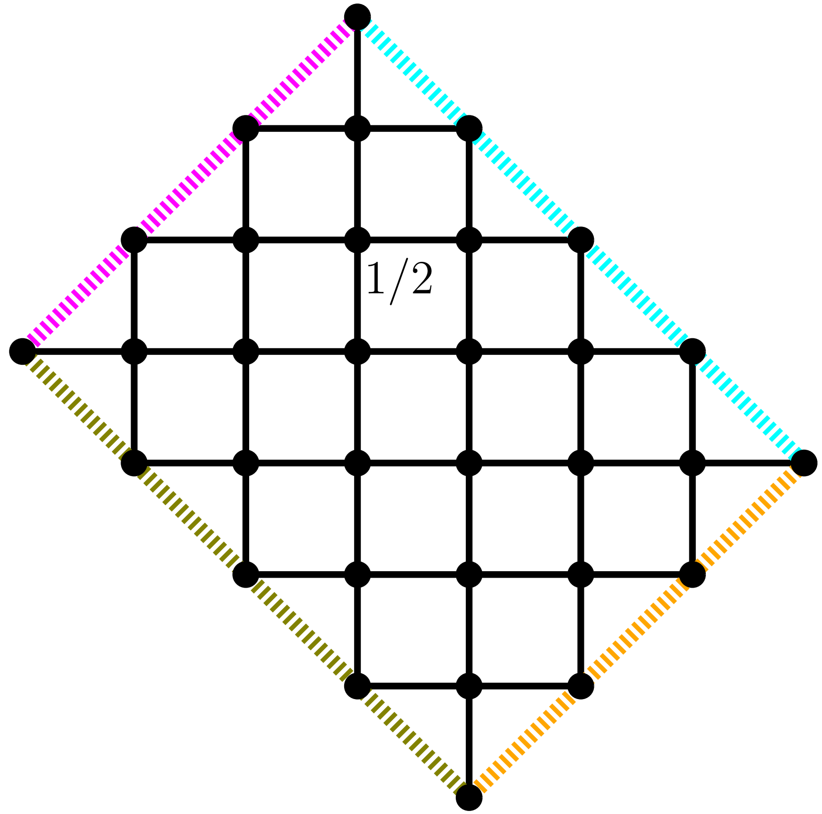




Two-commodity and weighted two-commodity tight spans

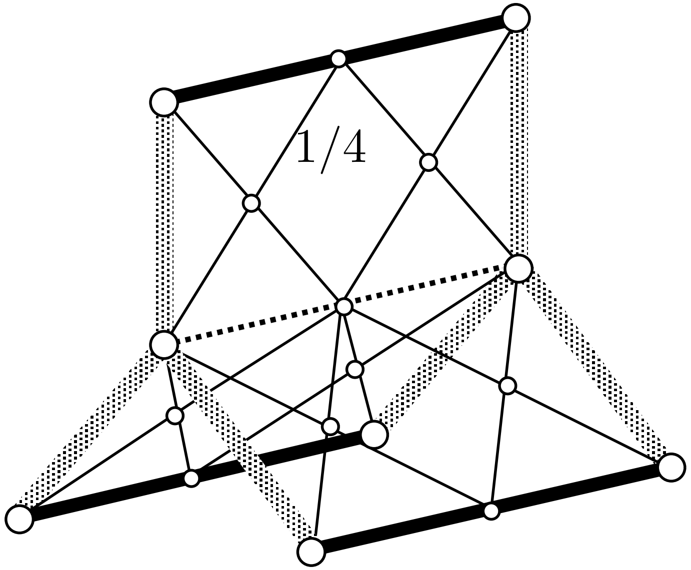


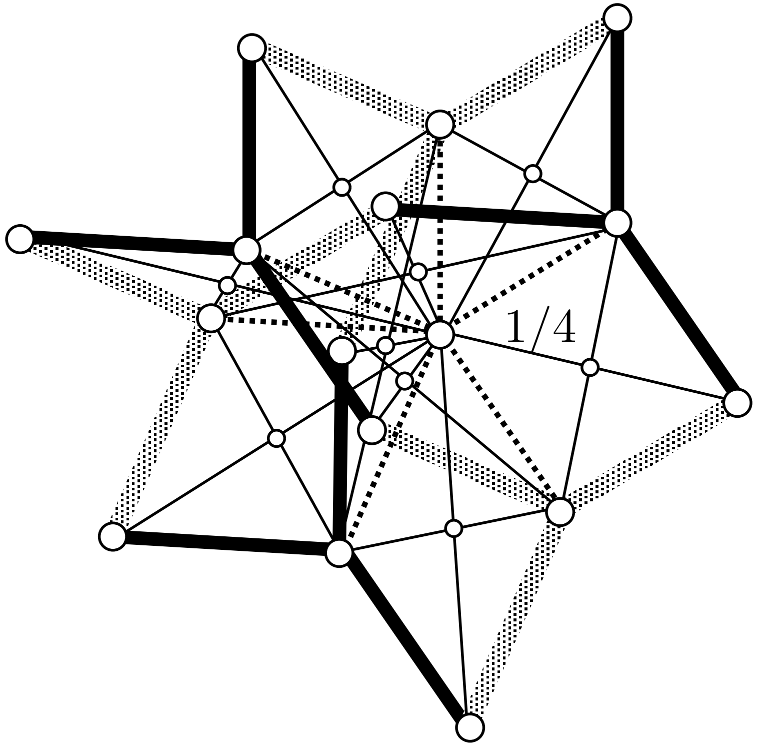
$(1, 1)$



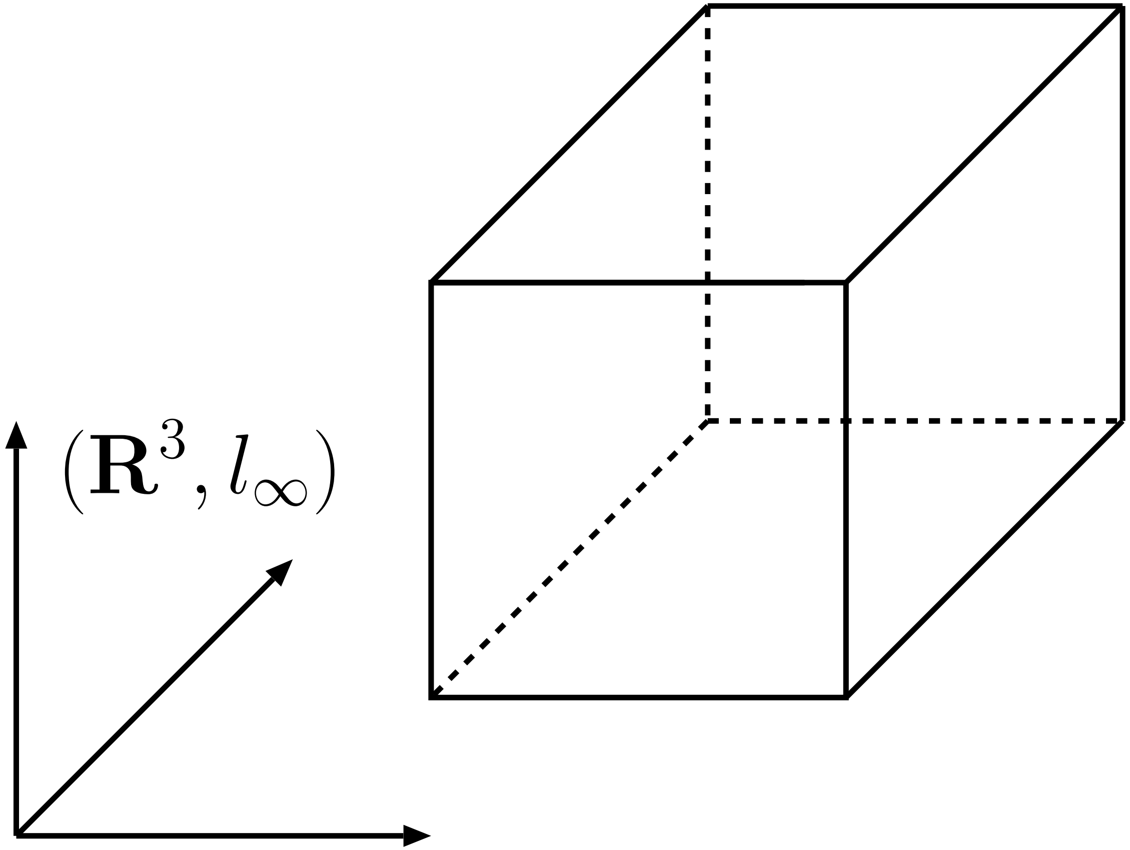
$(3, 4)$

$H = K_2 + K_3$ and $H = K_3 + K_3$





Three-commodity tight span ($H = K_2 + K_2 + K_2$)



Generalized Karzanov Conjecture (H. 08)

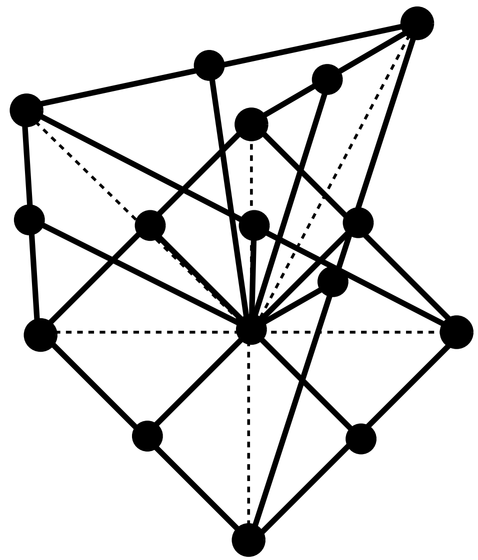
Conjecture (H. 08):
If $\dim T_\mu \leq 2$, then the fractionality of μ is finite(, and is one of $\{1, 2, 4\}$).

Remark (H.07): 0-1 weight $\mu \Rightarrow$ Karzanov conjecture.

The fractionality problem of flows is much more difficult than that of metrics !

Theorem (H. 08)

If μ is the graph metric of $K_{n,m}$ $(n, m \geq 3)$,
then the fractionality is finite $\in \{4, 8, 12, 24\}$.



$\Rightarrow$*the complete classification of demand graphs having finite fractionality in multiflow feasibility problems.*

Concluding remark

Multiflow combinatorial dualities $\Leftrightarrow$ 2-dimensionality of tight spans

Furture work

Splitting-off v.s. Geometry of tight spans (in preparation)

Historical remark

56 : Aronszajn and Panitchpakdi (hyperconvexity, injective space)

64 : Isbell (injective hull)

84 : Dress (tight span)

98 : Karzanov, Chepoi (applications to the multiflow theory)

06 : Hirai (nonmetric tight span)