Weakly modular graphs and nonpositive curvature

Hiroshi Hirai
The University of Tokyo

Joint work with J. Chalopin, V. Chepoi, and D. Osajda

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Contents

- Weakly modular graphs (Chepoi 89)
- Connections to nonpositively curved spaces
- Some results

J. Chalopin, V. Chepoi, H. Hirai, D. Osajda
Weakly modular graphs and nonpositive curvature
G is weakly modular (WM) ⇔

(TC) ∀ x, y, z : x ~ y, d(x, z) = d(y, z)
⇒ ∃ u: x ~ u ~ y, d(u, z) = d(x, z) - 1

(QC) ∀ x, y, w, z : x ~ w ~ y, d(w, z) - 1 = d(x, z) = d(y, z)
⇒ ∃ u: x ~ u ~ y, d(u, z) = d(x, z) - 1
G is weakly modular (WM) ⇔

(TC) ∀ x,y,z : x~y, d(x,z) = d(y,z)  
⇒ ∃u: x ~ u ~ y, d(u,z) = d(x,z) - 1

(QC) ∀ x,y,w,z : x~w~y, d(w,z)-1= d(x,z) = d(y,z)  
⇒ ∃u: x ~ u ~ y, d(u,z) = d(x,z) - 1
Classes of WM graph

modular graph = bipartite WM

orientable modular graph

dual polar space

modular semilattice

modular lattice

projective geometry

median graph

median semilattice

distributive lattice

Boolean lattice = cube

weakly bridged graph

tree
• Some classes of WM graphs are naturally associated with “metrized complex” of nonpositively-curvature-like property

• median graph ~~ CAT(0) cube complex
  (Gromov 87)

• bridged graph ~~ systolic complex
  (Haglund 03, Januszkiewicz-Swiantkowski 06)

• modular lattice ~~> orthoscheme complex
  (Brady-McCammond 10)
CAT(0) space

- geodesic metric space such that every geodesic triangle is “thin”

\[
d(p(t), z) \leq \| p'(t) - z' \|
\]

\[
d(y, z) = \| y' - z' \|
\]

\[
d(x, y) = \| x' - y' \|
\]

\[
d(z, x) = \| z' - x' \|
\]
Median graph

↔ every triple of vertices admits a unique median
↔ bipartite WM without K2,3

Median graph is obtained by “gluing” cubes
Median complex

:= cube complex obtained by filling “cube” to each cube-subgraph of median graph
Median complex

:= cube complex obtained by filling “cube” to each cube-subgraph of median graph

\[ \sim [0,1]^3 \]

\[ \sim [0,1]^2 \]
Median complex

:= cube complex obtained by filling “cube” to each cube-subgraph of median graph

Thm (Chepoi, 2000)
Median complex \(\equiv\) CAT(0) cube complex
c.f. Gromov’s characterization of CAT(0) cube complex
Folder complex

:= B2-complex obtained by filling “folder” to each $K_{2,m}$ subgraph of bipartite WM without $K_{3,3}$ and $K_{3,3}^\rightarrow$
Folder complex

:= B2-complex obtained by filling “folder” to each $K_{2,m}$ subgraph of bipartite WM without $K_{3,3}$ and $K_{3,3}^\sim$. 

![Diagram of folder complex]
Folder complex

:= B2-complex obtained by filling “folder” to each K2,m subgraph of bipartite WM without K3,3 and K3,3^-

Thm (Chepoi 2000)

Folder complex \equiv \text{CAT}(0) \text{ B2-complex}
Orthoscheme complex (Brady-McCammond10)

P: graded poset
K(P): = complex obtained by filling

to each maximal chain $x_0 < x_1 < \cdots < x_k$, $k=1,2,3..$
What are posets $P$ for which $K(P)$ is CAT(0)?
Conjecture (Brady-McCammond 10)
$K(P)$ is CAT(0) for modular lattice $P$. 
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$K(P)$ is CAT(0) for modular lattice $P$.

Theorem (Haettel, Kielak, and Schwer 13) 
$K(P)$ is CAT(0) for “complemented” modular lattice $P$. 
~ lattice of subspaces of vector space
Conjecture (Brady-McCammond 10)
$K(P)$ is CAT(0) for modular lattice $P$.

Theorem (Haettel, Kielak, and Schwer 13)
$K(P)$ is CAT(0) for “complemented” modular lattice $P$.
~ lattice of subspaces of vector space

Theorem (CCHO14)
$K(P)$ is CAT(0) for modular lattice $P$. 
Idea for proof

Obs: If $P$ is distributive, then $K(P) = \text{order polytope}$.

Thm [Birkhoff-Dedekind]
For two chains in modular lattice, there is a distributive sublattice containing them.

plus standard proof technique of
“spherical building is CAT(1)”
Conjecture [CCHO14]
K(P) is CAT(0) for modular semilattice P.

Modular semilattice
= semilattice whose covering graph is bipartite WM
Conjecture [CCHO14]
K(P) is CAT(0) for modular semilattice P.

Modular semilattice
= semilattice whose covering graph is bipartite WM

Median semilattice
= semilattice whose covering graph is median graph
Conjecture [CCHO14]
$K(P)$ is $\text{CAT}(0)$ for modular semilattice $P$.

Modular semilattice
$= \text{semilattice whose covering graph is bipartite WM}$

Median semilattice
$= \text{semilattice whose covering graph is median graph}$

Theorem [CCHO14]
$K(P)$ is $\text{CAT}(0)$ for median semilattice $P$.

$\leftarrow \text{Gluing construction (Reshetnyak’s gluing theorem)}$
We introduced a new class of WM graph, SWM graph
\[\text{SWM graph} \defeq \text{WM without K4}\sim \text{ and isometric K3,3}\sim\]
We introduced a new class of WM graph, 

\textbf{SWM graph}

:= WM without $K4^-$ and isometric $K3,3^-$

- affine building of type C
- orientable modular graph
- dual polar space
- modular semilattice
- modular lattice
- projective geometry
- median graph
- median semilattice
- distributive lattice
- tree

Boolean lattice = cube
Metrized complex $K(G)$ from SWM-graph $G$

$B(G) :=$ the set of all **Boolean-gated sets** of $G$

$X$: Boolean-gated $\iff$

- $x, y \in X$, $x \sim u \sim y \Rightarrow u \in X$,
- $x, y \in X$: $d(x, y) = 2 \Rightarrow \exists$ 4-cycle $\ni x, y$
Metrized complex $K(G)$ from SWM-graph $G$

$B(G) := \text{the set of all Boolean-gated sets of } G$

$X: \text{Boolean-gated } \iff$

$x, y \in X, x \sim u \sim y \Rightarrow u \in X,$

$x, y \in X: d(x, y) = 2 \Rightarrow \exists \text{ 4-cycle } \ni x, y$

$\rightarrow \text{Boolean-gated set induces dual polar space}$

$\rightarrow B(G): \text{graded poset w.r.t. (reverse) inclusion}$
Metrized complex $K(G)$ from SWM-graph $G$

$B(G) :=$ the set of all \textcolor{blue}{Boolean-gated sets} of $G$

$X$: Boolean-gated $\iff$

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$\rightarrow$ Boolean-gated set induces dual polar space

$\rightarrow B(G):$ graded poset w.r.t. (reverse) inclusion

$K(G) :=$ orthoscheme complex of $B(G)$
A graph $G$ with vertices $\{1, 2, 3, \ldots, 9\}$ and cliques $\{1,2,3\}, \{3,4\}, \{5,6\}, \ldots, \{8,9\}$. The graph $B(G)$ includes generalized quadrangles formed by the edges connecting cliques.
B(G)  

{1}, {2}, {3}, ..., {9}  

vertices

{1,2,3}, {3,4}, {5,6}, ..., {8,9}  

cliques

{3,4}  

generalized quadrangle
B(G)

{1}, {2}, {3}, \ldots, {9}  vertices

{1,2,3}, {3,4}, {5,6}, \ldots, {8,9}  cliques

generalized quadrangle

K(G)
G: median graph $\rightarrow$ B(G): set of cube-subgraphs
$\rightarrow$ K(G) subdivides median complex
G: median graph $\rightarrow$ B(G): set of cube-subgraphs
  $\rightarrow$ K(G) subdivides median complex

G: bipartite WM without K3,3 and K3,3$^-$
  $\rightarrow$ B(G): set of maximal K2,m subgraphs
  $\rightarrow$ K(G) subdivides folder complex
G: median graph $\rightarrow$ B(G): set of cube-subgraphs
$\rightarrow$ K(G) subdivides median complex

G: bipartite WM without K3,3 and K3,3^- $\rightarrow$ B(G): set of maximal K2,m subgraphs
$\rightarrow$ K(G) subdivides folder complex

G: SWM from affine building $\Delta$ of type C $\rightarrow$ K(G) = the standard metrication of $\Delta$
Conjecture (CCHO14)
$K(G)$ is CAT(0) for SWM-graph $G$. 
Some Topological Graph Theory result

Lemma (CCHO14)
Triangle-Square complex of WM-graph is simply-connected
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Lemma (CCHO14)
Triangle-Square complex of WM-graph is simply-connected
A Local-to-Global characterization of WM-graph
(analogue of Cartan-Hadamard theorem ?)

Theorem (CCHO14)
If G is locally-WM and TS-complex of G
is simply-connected, then G is WM.

Locally-WM: (TC) & (QC) with \( d(x, z) = d(y, z) = 2 \)
A Local-to-Grobal characterization of WM-graph (analogue of Cartan-Hadamard theorem ?)

Theorem (CCHO14)
If G is locally-WM and TS-complex of G is simply-connected, then G is WM.

Locally-WM: (TC) & (QC) with d(x,z) = d(y,z) = 2

Theorem (CCHO14)
The 1-skeleton of the universal cover of TS-complex of locally-WM-graph is WM.
Thank you for your attention!