Tree metrics and edge-disjoint S-paths

Hiroshi Hirai (U. Tokyo)
and
Gyula Pap (Eötvös U.)

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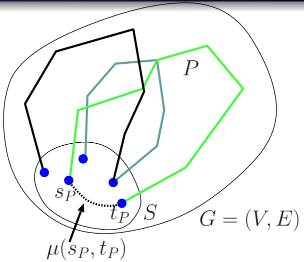
μ-weighted mincost edge-disjoint S-paths problem

$G = (V, E)$: an undirected graph
$S \subseteq V$: terminal set
$\mu : \binom{S}{2} \to \mathbf{Z}_+$: terminal weight
$a : E \to \mathbf{Z}_+$: edge-cost



Def: S-path $\overset{\text{def}}{\Longleftrightarrow}$ path connecting distinct terminals in S

Problem: μ-CEDP

$$\text{Maximize} \sum \{\mu(s_P, t_P) - a(P) \mid P \in \mathcal{P}\}$$

over all edge-disjoint set $\mathcal{P}$ of S-paths

μ-CEDP is NP-hard for almost μ ($\supseteq$ integer 2-commodity flow)
But ...

Mader's min-max formula: $(\mu, a) = (\mathbf{1}, \mathbf{0})$

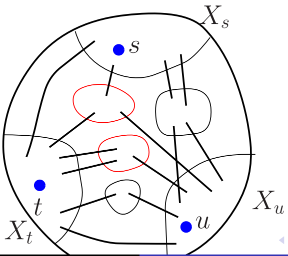
Def: $\mathrm{opt}(\mu; G, a) := \max_{\mathcal{P}} \sum_{P \in \mathcal{P}} \mu(s_P, t_P) - a(P)$

Def: $\delta^G X :=$ the set of edges joining X and $V \setminus X$ (cut)

Theorem (Mader 78)

$$\mathrm{opt}(\mathbf{1}; G, \mathbf{0}) = \frac{1}{2} \min_{\{X_s\}} \left\{ \sum_{s \in S} |\delta^G X_s| - \kappa \right\},$$

where $\{X_s\}_{s \in S}$: disjoint subsets with $s \in X_s$,
κ: the number of "odd" components of $G - \bigcup_{s \in S} X_s$.



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- Node-disjoint version (Mader 78),
- Linear matroid matching (Lovász 80, Schrijver 03),

$\Rightarrow (\mu, a) = (p\mathbf{1}, \mathbf{0}) \in P \ (\forall p > 0)$

Striking result (Karzanov 93, unpublished preprint 66pp)

$(p\mathbf{1}, *) \in P$

Tree metrics

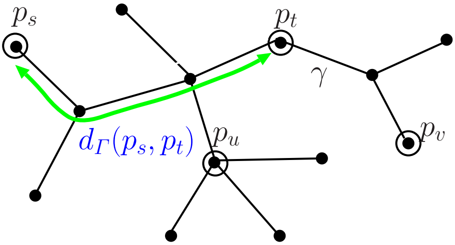
Question: *Does there exist other nontrivial weight* μ *for which* μ*-CEDP is tractable and has min-max* ?

Answer: Yes !

Definition

μ is a *tree metric* $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ tree Γ, $\{p_s\}_{s \in S} \subseteq V\Gamma$, $\gamma > 0$ s.t.

$$\mu(s,t) = \gamma d_\Gamma(p_s, p_t) \quad (s, t \in S).$$



Rem: **1** is a tree metric realized by a star with $\gamma = 1/2$.

Main result: min-max formula & polytime solvability

Def: G is *inner Eulerian* $\overset{\text{def}}{\Longleftrightarrow}$ $\forall x \in V \setminus S$ has even degree

Def: *inner-odd-join* $F \subseteq E$ $\overset{\text{def}}{\Longleftrightarrow}$ $G - F$ is inner Eulerian

$\Longleftrightarrow$ T-join in G/S with $T :=$ the set of odd nodes in G/S

Theorem (H & P, 2010)

If $\mu : \binom{S}{2} \to \mathbf{Z}_+$ is a tree metric realized by $(\Gamma, \{p_s\}_{s \in S}; 1/2)$, then

$$\mathrm{opt}(\mu; G, a) = \min_{\rho} \max_{F} \left(\frac{1}{2} d_\Gamma \circ \rho - a \right)(E \setminus F),$$

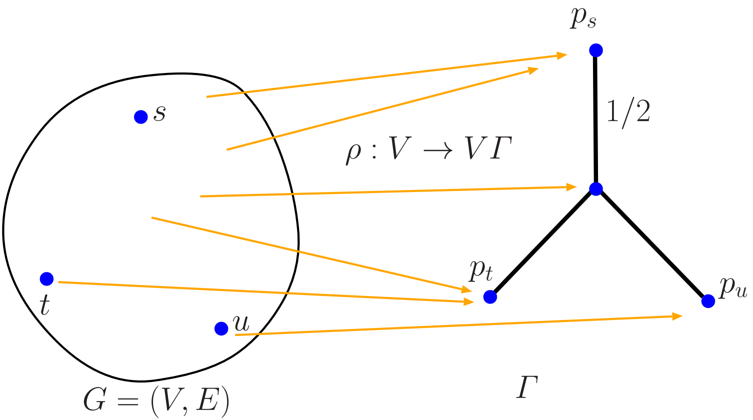
where $\rho : V \to V\Gamma$ with $\rho(s) = p_s$, F: inner-odd-join, and $\exists$ polytime algorithm.

$d_\Gamma \circ \rho : E \to \mathbf{R}_+ \overset{\text{def}}{\Longleftrightarrow} (d_\Gamma \circ \rho)(e) = d_\Gamma(\rho(x), \rho(y))$ $(e = xy \in E)$.

Rem: dual half-integrality (conjectured by Karzanov for $\mu = p\mathbf{1}$)

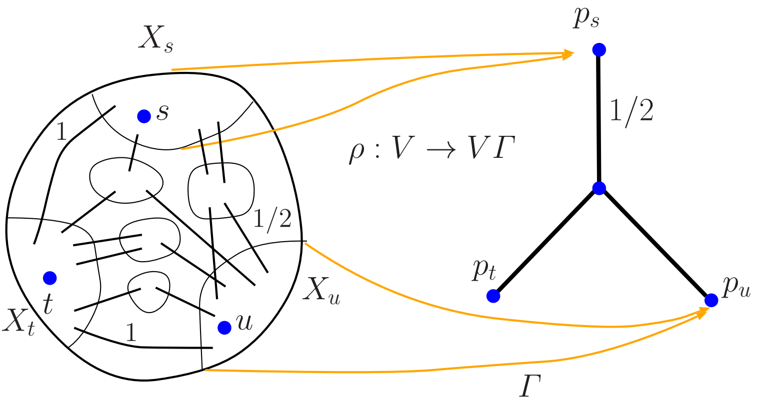
Example: $(\mu, a) = (\mathbf{1}, \mathbf{0})$

$$\mathrm{opt}(\mu; G, a) = \min_{\rho} \max_{F} \left(\frac{1}{2} d_{\Gamma} \circ \rho - a \right) (E \setminus F)$$



Example: $(\mu, a) = (\mathbf{1}, \mathbf{0})$

$$\mathrm{opt}(\mu; G, a) = \frac{1}{2}\left\{(d_\Gamma \circ \rho)(E) - \min_F (d_\Gamma \circ \rho)(F)\right\}$$



Example: $(\mu, a) = (\mathbf{1}, \mathbf{0})$

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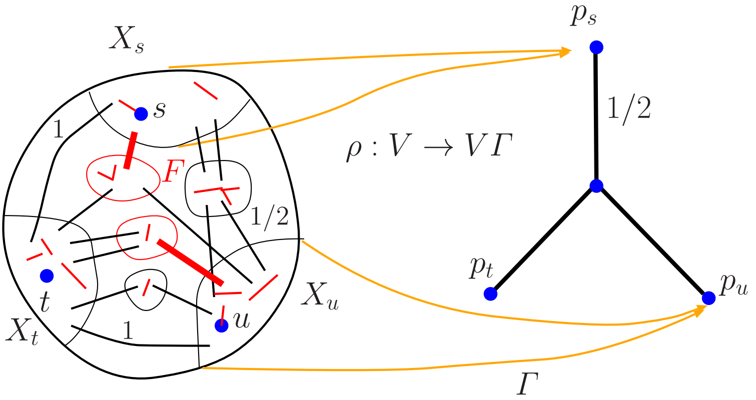

Main result: NP-hardness

Def: μ is a *truncated tree metric*
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ tree metric $\bar{\mu}$, $r : S \to \mathbf{Q}_+$ s.t.

$$\mu(s,t) = \max\{\bar{\mu}(s,t) - r(s) - r(t), 0\} \quad (s, t \in S).$$

$\Longleftrightarrow$ distance between *balls* with radius $r(\cdot)$ in a tree.

Rem: μ-CEDP $\Rightarrow$ $\bar{\mu}$-CEDP $\quad$ ($\to$ truncated tree metric $\in P$)

Theorem (H & P, 2010)

If μ is *not* a truncated tree metric,
then μ-CEDP is NP-hard (even if $a = \mathbf{0}$).

Corollary

Unless $P = NP$,
tractable class of weights = {truncated tree metrics}.

Application: approx. algorithm for "near" tree metrics

Problem: μ-EDP (cost-less version of μ-CEDP)

Maximize $\sum_{P \in \mathcal{P}} \mu(s_P, t_P)$ over edge-disjoint S-paths $\mathcal{P}$.

Current best: $O(\sqrt{|V|})$-approximation (Chekuri-Khanna-Shepherd 06)

Obs. *If $\mu^* \leq \mu \leq \alpha\mu^* \Rightarrow$ any optimal solution for μ^*-EDP is α-approximation for μ-EDP.*

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Theorem (Bădoiu-Indyk-Sidiropoulos, SODA 2007)

If μ is a graph metric with distortion α into tree metrics $\Rightarrow$
$\exists$ polytime algorithm finding a tree metric with distortion $O(\alpha)$.

Current best: 6α (Chepoi-Dragan-Newman-Rabinovich-Vaxe, APPROX 2010).

Corollary

If μ is a graph metric with distortion α into tree metrics $\Rightarrow$
$\exists$ $O(\alpha)$-approximation algorithm for μ-EDP.

Proof idea (min-max & polytime solvability)

Obs. $\mathrm{opt}(\mu; G, a) = \max_F \mathrm{opt}(\mu; G - F, \mathbf{0}) - a(E \setminus F)$

LP-relaxation

(Karzanov 93, Brunetta-Conforti-Fischetti 00, Keijsper-Pendavingh-Stougie 06)

$$\max \quad \sum_{P} \mu(s_P, t_P) f(P) - \sum_{e \in E} a(e)(1 - \xi(e))$$

$$\text{s.t.} \quad \sum_{P, e \in P} f(P) \leq 1 - \xi(e) \quad (e \in E)$$

$$\xi \in \text{inner-odd-join polytope}, \ f : \{S\text{-paths}\} \to \mathbf{R}_+$$

- we prove: if μ is tree metric $\Rightarrow$ this LP has an integral optimum and

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LP-dual

$$\begin{array}{ll} \min & \max_F (d - a)(E \setminus F) \\ \mathrm{s.t.} & d\text{: metric on } V, \\ & d(s,t) \geq \mu(s,t) \ \ s,t \in S. \end{array}$$

• we prove: if μ is tree metric $\Rightarrow$ this LP has an integral optimum and dual-LP is attained by $d = \frac{1}{2} d_\Gamma \circ \rho$.

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Obs. $\mathrm{opt}(\mu; G, a) = \max_F \mathrm{opt}(\mu; G - F, \mathbf{0}) - a(E \setminus F)$

LP-relaxation

$$\begin{aligned} \max \quad & \sum_P \mu(s_P, t_P) f(P) - \sum_{e \in E} a(e)(1 - \xi(e)) \\ \text{s.t.} \quad & \sum_{P, e \in P} f(P) \leq 1 - \xi(e) \quad (e \in E) \\ & \xi \in \text{inner-odd-join polytope}, \; f : \{S\text{-paths}\} \to \mathbf{R}_+ \end{aligned}$$

- we prove: if μ is tree metric $\Rightarrow$ this LP has an integral optimum and dual-LP is attained by $d = \frac{1}{2} d_\Gamma \circ \rho$.

- $\mathrm{opt}(\mu; G, a)$ is evaluated in polytime by *separation* $\equiv$ *optimization* (Grötschel-Lovász-Schrijver 81) $\Rightarrow$ splitting-off & edge-deletion technique.

Proof idea (hardness)
If $\mu \neq$ truncated tree metric $\Rightarrow$ μ-EDP is NP-hard

Theorem (classic result in 70's)

μ is a tree metric $\Leftrightarrow$ it is a metric, and $\forall\{s, t, u, v\}$,

$$\mu(s,t) + \mu(u,v) \leq \max\{\mu(s,u) + \mu(t,v), \mu(s,v) + \mu(t,u)\}.$$

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Theorem (H & P, 2010)

μ is a truncated tree metric $\Leftrightarrow$ $\forall\{s, t, u, v\}$

$$\mu(s,t) + \mu(u,v) \leq \max\left\{\begin{array}{l}\mu(s,t), \mu(u,v), \\ \mu(s,u) + \mu(t,v), \mu(s,v) + \mu(t,u)\end{array}\right\}.$$

For violating $\{s, t, u, v\} \subseteq S$,
$\mu|_{\{s,t,u,v\}}$-EDP solves a class of integer 2-commodity flow problems, known to be NP-hard (Even-Itai-Shamir 76).

Remarks & Future works

- Design of (simpler) combinatorial polytime algorithm
- Integrality gap of the LP-relaxation we used
- Node-disjoint version (Pap, coming soon ?)
- Gallai-Edmonds type structure theorem
  (extending Sebö-Szegö 2004)
- Discrete convexity
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Thank you for your attention !